EXTENDING THE SOLUTIONS AND THE EQUATIONS OF QUANTUM GRAVITY PAST THE BIG BANG SINGULARITY

CLAUS GERHARDT

Abstract. In [8] we recently proved that in our model of quantum gravity the solutions to the quantized version of the full Einstein equations or to the Wheeler-DeWitt equation could be expressed as products of spatial and temporal eigenfunctions, or eigendistributions, of self-adjoint operators acting in corresponding separable Hilbert spaces. Moreover, near the big bang singularity we derived sharp asymptotic estimates for the temporal eigenfunctions. In this paper we show that, by using these estimates, there exists a complete sequence of unitarily equivalent eigenfunctions which can be extended past the singularity by even or odd mirroring as sufficiently smooth functions such that the extended functions are solutions of the appropriately extended equations valid in $\mathbb{R}$ in the classical sense.

Contents

1. Introduction

A unified quantum theory incorporating the four fundamental forces of nature is one of the major open problems in physics. The Standard Model combines electromagnetism, the strong force and the weak force, but ignores gravity. The quantization of gravity is therefore a necessary first step to achieve a unified quantum theory.

Date: December 18, 2024.
2000 *Mathematics Subject Classification.* 83,83C,83C45.
Key words and phrases. quantization of gravity, quantum gravity, big bang singularity.

General relativity is a Lagrangian theory, i.e., the Einstein equations are derived as the Euler-Lagrange equation of the Einstein-Hilbert functional

$$\int_N (\bar R - 2\Lambda), \tag{1.1}$$

where $N = N^{n+1}$, $n \ge 3$, is a globally hyperbolic Lorentzian manifold, $\bar R$ the scalar curvature and Λ a cosmological constant. We also omitted the integration density in the integral. In order to apply a Hamiltonian description of general relativity, one usually defines a time function x^0 and considers the foliation of N given by the slices

$$M(t) = \{x^0 = t\}. \tag{1.2}$$

We may, without loss of generality, assume that the spacetime metric splits

$$d\bar s^2 = -w^2 (dx^0)^2 + g_{ij}(x^0, x) dx^i dx^j, \tag{1.3}$$

cf. [4, Theorem 3.2]. Then, the Einstein equations also split into a tangential part

$$G_{ij} + \Lambda g_{ij} = 0 \tag{1.4}$$

and a normal part

$$G_{\alpha\beta}\nu^\alpha\nu^\beta - \Lambda = 0, \tag{1.5}$$

where the naming refers to the given foliation. For the tangential Einstein equations one can define equivalent Hamilton equations due to the groundbreaking paper by Arnowitt, Deser and Misner [1]. The normal Einstein equations can be expressed by the so-called Hamilton condition

$$\mathcal H = 0, \tag{1.6}$$

where $\mathcal H$ is the Hamiltonian used in defining the Hamilton equations. In the canonical quantization of gravity the Hamiltonian is transformed to a partial differential operator of hyperbolic type $\hat{\mathcal H}$ and the possible quantum solutions of gravity are supposed to satisfy the so-called Wheeler-DeWitt equation

$$\hat{\mathcal H} u = 0 \tag{1.7}$$

in an appropriate setting, i.e., only the Hamilton condition (1.6) has been quantized, or equivalently, the normal Einstein equation, while the tangential Einstein equations have been ignored.

In [4] we solved the equation (1.7) in a fiber bundle E with base space $\mathcal S_0$,

$$\mathcal S_0 = \{x^0 = 0\} \equiv M(0), \tag{1.8}$$

and fibers $F(x)$, $x \in \mathcal S_0$,

$$F(x) \subset T^{0,2}_x(\mathcal S_0), \tag{1.9}$$

the elements of which are the positive definite symmetric tensors of order two, the Riemannian metrics in $\mathcal S_0$. The hyperbolic operator $\hat{\mathcal H}$ is then expressed in the form

$$\hat{\mathcal H} = -\Delta - (R - 2\Lambda)\varphi, \tag{1.10}$$

where Δ is the Laplacian of the DeWitt metric given in the fibers, R the scalar curvature of the metrics $g_{ij}(x) \in F(x)$, and φ is defined by

$$\varphi^2 = \frac{\det g_{ij}}{\det \rho_{ij}}, \tag{1.11}$$

where ρ_{ij} is a fixed metric in $\mathcal{S}_0$ such that instead of densities we are considering functions.

The Wheeler-DeWitt equation only represents the quantization of the normal Einstein equations and ignores the tangential Einstein equations. In order to quantize the full Einstein equations we incorporated the Hamilton condition into the right-hand side of the Hamilton equations to obtain a scalar evolution equation such that the Hamilton equations and this scalar evolution are equivalent to the full Einstein equations, cf. [8, Theorem 1.3.4, p. 12]. For the quantization of this evolution equation we defined the base space of the fiber bundle E to be the Cauchy hypersurface $(\mathcal{S}_0, \bar\sigma_{ij})$ of the quantized spacetime, where $\bar\sigma_{ij}$ is the induced metric. We also choose the metric ρ_{ij} in (1.11) to be equal to $\bar\sigma_{ij}$. The result of this quantization was a hyperbolic equation in E.

The fibers $F(x)$ over $x \in \mathcal{S}_0$ are Riemannian metrics $g_{ij}(x)$ if external fields are excluded. In an appropriate local trivialization we obtained a coordinate system (ξ^a), $0 \le a \le m$,

$$m = \frac{(n-1)(n+2)}{2},$$

$n = \dim \mathcal{S}_0$, such that the metrics g_{ij} can be written

$$g_{ij} = t^{\frac{4}{n}} \sigma_{ij},$$

where

$$0 < t = \xi^0 < \infty$$

and the metric σ_{ij} belongs to the hypersurface or subbundle

$$M = \{t = 1\} \subset E.$$

The solutions u then depend on the variables (t, σ_{ij}, x), where σ_{ij} does not depend on t and t not on x. We refer to t as quantum time and x, σ_{ij} as spatial variables.

In the papers [5, 7] we could express u as a product of eigenfunctions

$$u = w\hat v v, \tag{1.12}$$

where $w = w(t)$ is the temporal eigenfunction, $\hat v = \hat v(\sigma_{ij}(x))$ can be identified with an eigenfunction of the Laplacian of the symmetric space

$$X = SL(n, \mathbb{R})/SO(n) \tag{1.13}$$

such that

$$\hat v(\bar\sigma_{ij}(x)) = 1 \qquad \forall\, x \in \mathcal{S}_0, \tag{1.14}$$

where $\bar\sigma_{ij}$ is the fixed induced metric of $\mathcal{S}_0$. The eigenfunctions $\hat v$ represent the elementary gravitons corresponding to the degrees of freedom in choosing the entries of Riemannian metrics with determinants equal to one. These are all the degrees of freedom available because of the coordinate system invariance: For any smooth Riemannian metric there exists an atlas such that the determinant of the metric is equal to one, cf. [8, Lemma 3.2.1, p. 74]. The function v is an eigenfunction of an essentially self-adjoint differential operator in $\mathcal{S}_0$.

At first, the temporal eigenfunctions w were only the solutions of an ODE. Later, in [7, Section 5] we proved that they were the eigenfunctions of an essentially self-adjoint differential operator in $\mathbb{R}_+$, provided n is sufficiently large and $\Lambda < 0$ and the Cauchy hypersurface $(\mathcal{S}_0, \bar\sigma_{ij})$ is either a space of constant curvature like $\mathbb{R}^n$ and $\mathbb{H}^n$ or a metric product of the form

$$\mathcal{S}_0 = \mathbb{R}^{n_1} \times M_0, \tag{1.15}$$

where M_0 is a smooth, compact and connected manifold of dimension $n-n_1$,

$$\dim M_0 = n - n_1 = n_0, \tag{1.16}$$

and where

$$\bar\sigma = \delta \otimes g \tag{1.17}$$

is a metric product; δ is the standard Euclidean metric and g a Riemannian metric in M_0, cf. [7, Section 5].

But in [8, Chapter 4.2] we were able to prove this property for arbitrary $n \ge 3$ and $\Lambda < 0$ and, in case $n = 3$, even for $\Lambda > 0$ by introducing an additional scalar fields map in the action functional, i.e., a map

$$\Phi : N \to \mathbb{R}^k, \tag{1.18}$$

where $N = I \times \mathcal{S}_0$ is the original spacetime which is to be quantized. Let $(\bar g_{\alpha\beta})$ be the Lorentzian metric in N, the scalar field Lagrangian is defined by

$$L_S = -\frac12 \bar g^{\alpha\beta}\gamma_{ab}\Phi^a_\alpha\Phi^b_\beta\sqrt{|\bar g|}, \tag{1.19}$$

i.e., without a zero order term, (γ_{ab}) is the Euclidean metric in $\mathbb{R}^k$.

The temporal eigenfunctions w then have to satisfy the ODE

$$\begin{aligned} \frac{n}{16(n-1)} t^{-(m+k)}\frac{\partial}{\partial t}\big(t^{(m+k)}\frac{\partial w}{\partial t}\big) + t^{-2}(|\lambda|^2 + \rho^2 - \frac12|\theta_0|^2)w \\ + t^{2-\frac4n}\{(n-1)|\xi|^2 + \bar\mu_l\}w + (n-2)t^2\Lambda w = 0 \end{aligned} \tag{1.20}$$

in $0 < t < \infty$, where

$$|\lambda|^2 + \rho^2 \tag{1.21}$$

is an eigenvalue of an elementary graviton,

$$|\theta_0|^2, \tag{1.22}$$

an eigenvalue of $-\Delta_{\mathbb{R}^k}$ and

$$(n-1)|\xi|^2 + \bar\mu_l \tag{1.23}$$

with $\xi \in \mathbb{R}^{n_1}$ an eigenvalue of the spatial self-adjoint operator acting in (1.15).

Using the abbreviations

$$\mu_0 = \frac{16(n-1)}{n}(|\lambda|^2 + |\rho|^2 - \frac{1}{2}|\theta_0|^2), \tag{1.24}$$

$$m_1 = \frac{16(n-1)}{n}\{(n-1)|\xi|^2 + \bar{\mu}_l\} \tag{1.25}$$

and

$$m_2 = \frac{16(n-1)(n-2)}{n} \tag{1.26}$$

we can rewrite the equation (1.20) in the form

$$t^{-(m+k)}\frac{\partial}{\partial t}\big(t^{(m+k)}\frac{\partial w}{\partial t}\big) + t^{-2}\mu_0 w + t^{2-\frac{4}{n}} m_1 w + t^2 m_2 \Lambda w = 0. \tag{1.27}$$

This equation can be treated as an eigenvalue equation provided

$$\bar{\mu} = \mu_0 - \frac{(m+k-1)^2}{4} < 0. \tag{1.28}$$

Let us recall that

$$m = \frac{(n-1)(n+2)}{2}. \tag{1.29}$$

and

$$\rho^2 = \frac{(n-1)^2 n}{12}. \tag{1.30}$$

There are two ways how to treat (1.27) as an eigenvalue equation: First, the cosmological constant Λ, or better $-\Lambda$ can be looked at as an implicit eigenvalue, or secondly, if we consider $\Lambda < 0$ to be fixed, we could try to solve the eigenvalue problem

$$-t^{-(m+k)}\frac{\partial}{\partial t}\big(t^{(m+k)}\frac{\partial w}{\partial t}\big) - t^{-2}\mu_0 w - t^2 m_2 \Lambda w = \lambda t^{2-\frac{4}{n}} w \tag{1.31}$$

in $(0,\infty)$, where $\lambda > 0$ is a yet unknown eigenvalue such that λ would be equal to the spatial eigenvalue, i.e.,

$$\lambda = m_1 = \frac{16(n-1)}{n}\{(n-1)|\xi|^2 + \bar{\mu}_l\}. \tag{1.32}$$

In this case the corresponding eigenfunction w would be a solution of (1.27), i.e., it would be a temporal eigenfunction of our model of quantum gravity. We solved the implicit as well as the explicit eigenvalue problem in [8, Chapter 4] by choosing k in (1.28) sufficiently large such that $\bar{\mu} < 0$.

Since μ_0 is in general positive, unless we choose $|\theta_0|$ large which is not always possible or desirable, we considered the orthogonally equivalent function

$$u = t^{\frac{m+k-1}{2}} w \tag{1.33}$$

which satisfies the equation

$$-t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big)-t^{-2}\bar\mu u+t^2m_2^2u=\lambda t^{2-\frac4n}u, \tag{1.34}$$

where

$$\bar\mu=\mu_0-\left(\frac{m+k-1}{2}\right)^2 \tag{1.35}$$

which is negative if $k\in\mathbb{N}$ is large enough.

In [8, Theorem 3.4.9, p. 86] we proved

Theorem 1.1. *Let $u\in\mathcal{H}_2$ satisfy the equation (1.34) which we express in the form*

$$A_1u=-t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big)+t^{-2}\mu^2u+t^2m_2^2u=\lambda t^{2-\frac4n}u, \tag{1.36}$$

where the constants μ, m_2 and λ are strictly positive. Since μ is especially important, let us emphasize that

$$\mu^2=-\bar\mu=\frac{(m+k-1)^2}{4}-\mu_0 \tag{1.37}$$

and $\mu_0>0$. Then, there exists $0<t_0<1$ and positive constants p, c_1, c_2 such that u does not vanish in the interval $(0,t_0]$ and can be estimates by

$$c_1t^p\le|u(t)|\le c_2t^\mu\qquad\forall\, t\in(0,t_0], \tag{1.38}$$

where p,

$$\mu<p<\frac{m+k-1}{2}, \tag{1.39}$$

is arbitrary but fixed.

Here, we adapted the wording slightly to reflect the present assumptions, cf. [8, Theorem 4.2.4, p. 118].

If we combine gravity with the forces of the Standard Model then we cannot quantize the full Einstein equations but only the normal Einstein equation, i.e., the Hamilton condition. As a result we obtain the Wheeler-DeWitt equation which again can be solved by a product of spatial and temporal eigenfunctions or eigendistributions. in this case the temporal eigenfunction equation has the form, after using the same ansatz as before,

$$-t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big)-t^{-2}\bar\mu u+t^2m_2^2u=\lambda t^{-\frac23}u, \tag{1.40}$$

where

$$\bar\mu=\mu_0-\left(\frac{m+k-1}{2}\right)^2. \tag{1.41}$$

Comparing this equation with equation (1.34) there are two differences: First, the term μ_0 does not depend on $|\theta_0|$

$$\mu_0 = \frac{16(n-1)}{n}(|\lambda|^2 + |\rho|^2) \tag{1.42}$$

since we had to choose $\theta_0 = 0$, and secondly, the exponent of t on the right-side is $-\frac{2}{3}$. The first difference implies that only by requiring k to be large we could enforce $\bar{\mu} < 0$ and the negative exponent that the estimate (1.38) is slightly worse, but still good enough for our purpose. Indeed, we proved in [8, Theorem 5.5.5, p. 145]

Theorem 1.2. *Let $u \in \mathcal{H}_2$ satisfy the equation*

$$A_1 u = -t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big) + t^{-2}\mu^2 u + t^2 m_2^2 u = \lambda t^{-\frac{2}{3}} u, \tag{1.43}$$

where the constants μ, m_2 and λ are strictly positive. Since μ is especially important, let us emphasize that

$$\mu^2 = -\bar{\mu} = \frac{(m+k-1)^2}{4} - \mu_0 \tag{1.44}$$

and $\mu_0 > 0$. Then, for any small $\epsilon_0 > 0$, there exist $0 < t_0 < 1$ and positive constants p, c_1, c_2 such that u does not vanish in the interval $(0, t_0]$ and can be estimated by

$$c_1 t^p \le |u(t)| \le c_2 t^{\mu - \epsilon_0} \qquad \forall\, t \in (0, t_0], \tag{1.45}$$

where p,

$$\mu < p < \frac{m+k-1}{2}, \tag{1.46}$$

is arbitrary but fixed.

The eigenvalue equations (1.36) and (1.43) in the Hilbert space $\mathcal{H}_2$ can both be solved by complete sequences of mutually orthogonal eigenfunctions u_i with corresponding positive eigenvalues λ_i of multiplicity one satisfying

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1.47}$$

and

$$\lim_{i \to \infty} \lambda_i = \infty. \tag{1.48}$$

For a proof see [8, Theorem 3.4.5, p. 84] and Theorem 3.10 on page 21, where we shall prove a corresponding result for a more general right-hand side which includes our two cases.

As a corollary, which we like to formulate as a theorem, we deduce:

Theorem 1.3. *Let $w_i \in \hat{\mathcal{H}}_2$ be related to a function u_i by*

$$w_i = t^{-\frac{m+k-1}{2}} u_i \tag{1.49}$$

and assume that $u_i \in \mathcal{H}_2$ *satisfies an equation of the form*

$$A_1 u = -t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big) + t^{-2}\mu^2 u + t^2 m_2^2 u = \lambda t^{-\frac{2}{3}} u, \tag{1.50}$$

where the constants μ, m_2 *and* λ *are strictly positive and* μ *is defined by*

$$\mu^2 = -\bar{\mu} = \frac{(m+k-1)^2}{4} - \mu_0 \tag{1.51}$$

and $\mu_0 > 0$*. Then, for any small* $\epsilon_0 > 0$ *there exists* $0 < t_0 < 1$ *and positive constants* p, c_1, c_2*, such that* w_i *does not vanish in the interval* $(0, t_0]$ *and can be estimates by*

$$c_1 t^{p-\frac{m+k-1}{2}} \le |w_i(t)| \le c_2 t^{\mu-\epsilon_0-\frac{m+k-1}{2}} \qquad \forall\, t \in (0, t_0], \tag{1.52}$$

where p*,*

$$\mu < p < \frac{m+k-1}{2}, \tag{1.53}$$

is arbitrary but fixed. Hence, we conclude

$$\lim_{t\to 0} |w_i(t)| = \infty. \tag{1.54}$$

The eigenfunctions w_i in the previous theorem are the solutions of the original temporal eigenfunctions equation and they are the eigenfunctions of a self-adjoint operator in Hilbert space. The u_i are the unitarily equivalent eigenfunctions of a unitarily equivalent self-adjoint operator. In Section 3 on page 13 we shall show that the unitarily equivalent eigenfunctions

$$\tilde{u}_i = t^{\frac{1}{2}} u_i \tag{1.55}$$

can be extended past the singularity by an even reflection as sufficiently smooth functions provided the coefficient μ^2 in (1.44) is large enough. More precisely, we shall prove:

Theorem 1.4. *Let* $2 \le m_0 \in \mathbb{N}$ *be arbitrary and assume*

$$\mu + \frac{1}{2} > m_0, \tag{1.56}$$

then

$$\tilde{u}_i \in C^{m_0}([0, t_0]) \quad \wedge \quad \tilde{u}_i^{(m_0)}(0) = 0 = \lim_{t\to 0} \tilde{u}_i^{(m_0)}(t) \tag{1.57}$$

as well as

$$\lim_{t\to 0} \frac{\tilde{u}_i^{(k)}(t)}{t^{m_0-k}} = 0 \qquad \forall\, 1 \le k \le m_0,\ k \in \mathbb{N}, \tag{1.58}$$

where $\tilde{u}_i^{(k)}$ *denotes the* k*-th derivative of* $\tilde{u}_i$*. These properties are also valid for the extended functions.*

Furthermore, we shall prove

Corollary 1.5. *If the assumption of the preceding theorem is satisfied then the extended solutions $\tilde{u}_i$ also satisfy the extended equations*

$$-\ddot{\tilde{u}}_i + t^{-2}\tilde{\mu}^2\tilde{u}_i + t^2 m_2^2\tilde{u}_i = \lambda_i|t|^q\tilde{u} \tag{1.59}$$

in $\mathbb{R}$, where we have to replace t^q by $|t|^q$ for obvious reasons. Let us emphasize that the lower order coefficients of the ODE exhibit a singularity in $t = 0$ but that both sides of the equation are continuous in the interval $(-\infty, \infty)$ and vanish in $t = 0$.

Here, the exponent q is any real number satisfying

$$-2 < q < 2. \tag{1.60}$$

2. The equations of quantum gravity

The tangential Einstein equations are equivalent to the Hamilton equations and the normal Einstein equation is equivalent to the Hamilton condition. By quantizing the Hamilton condition we obtain the Wheeler-DeWitt equation while ignoring the tangential Einstein equations. In order to quantize the full Einstein equations we consider the second Hamilton equations

$$\dot{\pi}^{ij} = -\frac{\delta H}{\delta g_{ij}}, \tag{2.1}$$

where

$$H = H_0 + H_1 \tag{2.2}$$

is the combined Hamilton function of the gravitational Hamiltonian H_0 and the scalar field map Hamiltonian H_1. Thus, we infer

$$g_{ij}\dot{\pi}^{ij} = -g_{ij}\frac{\delta H}{\delta g_{ij}} = -g_{ij}\frac{\delta(H_0 + H_1)}{\delta g_{ij}}. \tag{2.3}$$

On the right-hand side of this evolution equation we then implement the Hamilton condition $H = 0$ in the form

$$pH = 0, \tag{2.4}$$

where $0 \neq p \in \mathbb{R}$ is an arbitrary real number to be determined later. After the quantization of the modified evolution equation (2.3) we obtain the hyperbolic equation

$$\begin{aligned}
&(\frac{n}{2} - 2 - p)\{-\frac{n}{16(n-1)}t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{u}) \\
&\quad + t^{-2}\Delta_M u + \frac{1}{2}t^{-2}\Delta_{\mathbb{R}^k}u\} - (n-1)t^{2-\frac{4}{n}}\tilde{\Delta}_\sigma u \\
&\quad - pt^{2-\frac{4}{n}}R_\sigma u + 2p\Lambda u + t^{-2}\Delta_{\mathbb{R}^k}u + pC_1 u = 0.
\end{aligned} \tag{2.5}$$

The preceding equation is evaluated at $(x, t, \sigma_{ij}, \theta^a)$, where $x \in \mathcal{S}_0$, $t \in \mathbb{R}_+$, $\sigma_{ij} \in M$ is the induced metric of a Cauchy hypersurface of the quantized globally hyperbolic spacetime and $\theta = \theta(x)$ is a coordinate in the fiber $\mathbb{R}^k$.

Let us recall that after quantization the components Φ^a of the scalar field are equal to the coordinates θ^a in $\mathbb{R}^k$ such that

$$\Phi^a(x) = \theta^a(x) \qquad \forall\, x \in \mathcal{S}_0 \tag{2.6}$$

and

$$C_1 = \frac{1}{2} t^{2-\frac{4}{n}} \sigma^{ij} \gamma_{ab} \theta^a_i \theta^b_j. \tag{2.7}$$

Since we only introduced the scalar field in order to prove that the temporal "eigenfunctions" are indeed eigenfunctions of a self-adjoint operator with a pure point spectrum we can simplify the left-hand side of (2.5) by choosing

$$\theta^a(x) = 1 \qquad \forall\, x \in \mathcal{S}_0,\ \forall\, 1 \le a \le k. \tag{2.8}$$

Hence, we have to solve the equation

$$\begin{aligned}
&(\frac{n}{2} - 2 - p)\{-\frac{n}{16(n-1)} t^{-(m+k)} \frac{\partial}{\partial t}(t^{(m+k)}\dot u) \\
&\quad + t^{-2}\Delta_M u + \frac{1}{2} t^{-2} \Delta_{\mathbb{R}^k} u\} - (n-1) t^{2-\frac{4}{n}} \tilde\Delta_\sigma u \\
&\quad - p t^{2-\frac{4}{n}} R_\sigma u + 2p t^2 \Lambda u + t^{-2} \Delta_{\mathbb{R}^k} u = 0,
\end{aligned} \tag{2.9}$$

where u depends on $(x, t, \sigma_{ij}, \theta^a)$. The parameter $p \in \mathbb{R}$, $p \ne 0$, is not yet specified.

As mentioned before the solution u should be a product of spatial and temporal eigenfunctions. In order to ensure that the temporal eigenfunctions are eigenfunctions of a self-adjoint operator we have to distinguish three cases:

Case 1: $\Lambda < 0$ and $n \ge 3$.

Then we choose

$$p = \frac{n}{2} - 1 \tag{2.10}$$

and consider the equation

$$\begin{aligned}
&\frac{n}{16(n-1)} t^{-(m+k)} \frac{\partial}{\partial t}(t^{(m+k)}\dot u) \\
&\quad - t^{-2}\Delta_M u + \frac{1}{2} t^{-2} \Delta_{\mathbb{R}^k} u - (n-1) t^{2-\frac{4}{n}} \tilde\Delta_\sigma u \\
&\quad - (\frac{n}{2} - 1) t^{2-\frac{4}{n}} R_\sigma u + (n-2) t^2 \Lambda u = 0.
\end{aligned} \tag{2.11}$$

Case 2: $\Lambda > 0$ and $n \ge 5$.

Then, we choose

$$p = \frac{n}{2} - 2 - \frac{1}{4} > 0 \tag{2.12}$$

and consider the equation

$$\begin{aligned}
&-\frac{1}{4}\frac{n}{16(n-1)}t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{u})\\
&+\frac{1}{4}t^{-2}\Delta_M u+\frac{9}{8}t^{-2}\Delta_{\mathbb{R}^k}u-(n-1)t^{2-\frac{4}{n}}\tilde{\Delta}_\sigma u\\
&-(\frac{n}{2}-\frac{9}{4})t^{2-\frac{4}{n}}R_\sigma u+(n-\frac{9}{2})t^2\Lambda u=0.
\end{aligned}\tag{2.13}$$

Case 3: $\Lambda>0$ and $n=3$.

Then we choose

$$p=-\frac{1}{4}\tag{2.14}$$

yielding

$$\begin{aligned}
&\frac{1}{4}\frac{n}{16(n-1)}t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{u})\\
&-\frac{1}{4}t^{-2}\Delta_M u+\frac{7}{8}t^{-2}\Delta_{\mathbb{R}^k}u-(n-1)t^{2-\frac{4}{n}}\tilde{\Delta}_\sigma u\\
&+\frac{1}{4}t^{2-\frac{4}{n}}R_\sigma u-\frac{1}{2}t^2\Lambda u=0.
\end{aligned}\tag{2.15}$$

For a more detailed exposition we refer to [8, Chapter 4.2].

Finally, let us look at the Wheeler-DeWitt equation which we solved when we quantized gravity combined with the forces of the Standard Model, cf. [6]. For our purpose the reference [8, Chapter 5.4] is more suitable since, there, we also added a scalar field map such that the combined Hamilton function has the form

$$\begin{aligned}
\mathcal{H}&=\mathcal{H}_G+\mathcal{H}_S+\mathcal{H}_{YM}+\mathcal{H}_H+\mathcal{H}_D\\
&=\mathcal{H}_G+\mathcal{H}_S+t^{-\frac{2}{3}}(\tilde{\mathcal{H}}_{YM}+\tilde{\mathcal{H}}_H+\tilde{\mathcal{H}}_D)\\
&\equiv\mathcal{H}_G+\mathcal{H}_S+t^{-\frac{2}{3}}\tilde{\mathcal{H}}_{SM},
\end{aligned}\tag{2.16}$$

where the subscripts YM, H, D refer to the Yang-Mills, Higgs and Dirac fields and SM to the fields of the Standard Model or to a corresponding subset of fields. The Hamilton constraint

$$\mathcal{H}=0\tag{2.17}$$

will be quantized by first quantizing the Hamiltonians $\mathcal{H}_G+\mathcal{H}_S$ in the fibers for general metrics resulting in a hyperbolic operator

$$\hat{\mathcal{H}}_G+\hat{\mathcal{H}}_S\tag{2.18}$$

But the expression

$$\hat{\mathcal{H}}_G u+\hat{\mathcal{H}}_S u\tag{2.19}$$

will be evaluated $(x,t,\delta_{ij},\bar{\theta}^a)$, where δ_{ij} is the standard Euclidean metric in $\mathcal{S}_0=\mathbb{R}^n$, $n=3$, and

$$\bar{\theta}^a(x)=1\qquad\forall\,1\le a\le k.\tag{2.20}$$

The Hamilton function $\mathcal{H}_{SM}$, which represents spatial fields and is independent of t, is quantized in $(\mathcal{S}_0, \delta_{ij})$ by the usual methods of Quantum Field Theory (QFT). The Wheeler-DeWitt equation then has the form

$$\begin{aligned}\hat{\mathcal{H}}u = \alpha_N^{-1}\{&\frac{n}{16(n-1)}t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{u})\\ &- t^{-2}\varDelta_M u\} + \alpha_N^{-1}2t^2\varLambda u + t^{-\frac{2}{3}}\hat{\mathcal{H}}_{SM}u = 0,\end{aligned} \tag{2.21}$$

where α_N is a positive coupling constant and where we also assume that u does not depend on $\theta^a(x)$.

We then solve the Wheeler-DeWitt equation by using separation of variables. The operator $\hat{\mathcal{H}}_{SM}$ is acting only in the base space $\mathcal{S}_0$, such that the spatial eigendistributions, or approximate eigendistributions, ψ satisfying

$$\hat{\hat{\mathcal{H}}}_{SM}\psi = \mu\psi, \qquad \mu > 0 \tag{2.22}$$

can be derived by applying standard methods of QFT.

The remaining operator in (2.21) is acting only in the fibers, i.e., we can use the eigenfunctions $v = v(\sigma_{ij})$ of $-\varDelta_M$, which represent the elementary gravitons, satisfying

$$-\varDelta_M v = (|\lambda|^2 + |\rho|^2)v \qquad \forall\,\sigma_{ij} \in M \tag{2.23}$$

and

$$v(\delta_{ij}) = 1 \qquad \forall\, x \in \mathcal{S}_0, \tag{2.24}$$

cf. [8, Theorem 3.2.3, p. 76], and where

$$|\rho|^2 = 1 \tag{2.25}$$

if $n = 3$, compare [8, equation (2.2.34), p. 49] and (1.30) on page 5.

Hence, we make the ansatz

$$u = wv\psi, \tag{2.26}$$

where $w = w(t)$ only depends on $t > 0$. Then, combining (2.21), (2.22), (2.23), (2.24) and (2.25) we derive an ODE which must be solved by w, namely,

$$\begin{aligned}&\frac{n}{16(n-1)}t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{w})\\ &\qquad + t^{-2}(|\lambda|^2+1)w + 2t^2\varLambda w + \alpha_N t^{-\frac{2}{3}}\mu w = 0.\end{aligned} \tag{2.27}$$

Rewriting this ODE as

$$-t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{w}) - \mu_0 t^{-2}w - m_2 t^2 \varLambda w = m_1 t^{-\frac{2}{3}}w, \tag{2.28}$$

where

$$\mu_0 = \frac{16(n-1)}{n}(|\lambda|^2+1), \tag{2.29}$$

$$m_1 = \frac{16(n-1)}{n}\alpha_N\mu \tag{2.30}$$

and

$$m_2 = \frac{32(n-1)}{n}, \tag{2.31}$$

then the left-hand side of (2.28) is identical to the left-hand side of equation (1.31) on page 5. However, on the right-hand side of these equations we have different powers of t which will lead to slightly different asymptotic estimates from above near the origin for the corresponding solutions. In order to unify the approach we shall consider the temporal equation

$$-t^{-(m+k)}\frac{\partial}{\partial t}(t^{(m+k)}\dot{w}) - \mu_0 t^{-2}w - m_2 t^2 \Lambda w = m_1 t^q w, \tag{2.32}$$

where

$$-2 < q < 2 \tag{2.33}$$

such that the resulting estimates can be applied in both cases.

Using the same transformation as in (1.33) on page 5 we define the function

$$u = t^{\frac{m+k-1}{2}} w \tag{2.34}$$

which satisfies the equation

$$-t^{-1}\frac{\partial}{\partial t}(t\frac{\partial u}{\partial t}) - t^{-2}\bar{\mu}u - t^2 m_2 \Lambda u = m_1 t^q u, \tag{2.35}$$

where

$$\bar{\mu} = \mu_0 - \left(\frac{m+k-1}{2}\right)^2 \tag{2.36}$$

is negative if $k \in \mathbb{N}$ is large enough. If in addition the cosmological constant is also negative

$$\Lambda < 0, \tag{2.37}$$

then (2.35) can be looked at as an eigenvalue equation with positive eigenvalues m_1 in an appropriate Hilbert space. We shall solve the eigenvalue problem in the next section and prove asymptotic estimates near the singularity which will allow us to deduce that unitarily equivalent eigenfunctions can be extended past the singularity as sufficiently smooth functions.

3. Extending the temporal solutions past the singularity

In this section we shall prove asymptotic estimates from above near the singularity for the solutions of the equation (2.35) and we shall use these estimates to conclude that the unitarily equivalent eigenfunction

$$\tilde{u} = t^{\frac{1}{2}} u \tag{3.1}$$

can be extended past the singularity under suitable assumptions.

The extension itself is fairly easy we simply mirror the solution on the positive axes to the negative axes where even or odd mirroring are both possible. The crucial point is to show that the mirrored functions are sufficiently smooth in $\mathbb{R}$ and that the temporal equation is valid in the classical sense even at the singularity $t = 0$. In order to achieve these results we have to prove that the temporal solutions and there derivatives, up to the order two at least, vanish sufficiently fast at $t = 0$.

Next, let us prove sharp estimates near the origin for eigenfunctions of the equation (2.35) which will play a fundamental role in deducing that the unitarily equivalent temporal eigenfunctions $\tilde{u}$ in (3.1) which are the eigenfunctions of unitarily equivalent self-adjoint operator, can be smoothly extended past the big bang singularity in $t = 0$.

For a better understanding we first need a few definitions. The operator

$$Bu = -t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big) + t^{-2}\mu^2 u \tag{3.2}$$

is known as a Bessel operator.

Definition 3.1. Let $I = (0, \infty)$ and let $r \in \mathbb{R}$. Then we define

$$L^2(I, r) = \{\, u \in L^2_{\text{loc}}(I, \mathbb{R}) \colon \int_I t^r |u|^2 < \infty \,\}. \tag{3.3}$$

$L^2(I, r)$ is a Hilbert space with scalar product

$$\langle u_1, u_2\rangle_r = \int_I t^r u_1 u_2, \tag{3.4}$$

but let us emphasize that we shall apply this definition only for $r \neq 2$. The scalar product $\langle\cdot,\cdot\rangle_2$ will be defined differently.

We consider real valued functions for simplicity but we could just as well allow complex valued functions with the standard scalar product, or more precisely, sesquilinear form.

Definition 3.2. For functions $u \in C^\infty_c(I)$ define the operator

$$A_1 u = -t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big) + t^{-2}\mu^2 u - t^2 m_2 \Lambda u, \tag{3.5}$$

as well as the scalar product

$$\langle u_1, u_2\rangle_2 = \langle Bu_1 + t^2 m_2 u_1, u_2\rangle_1 \qquad \forall\, u_1, u_2 \in C^\infty_c(I). \tag{3.6}$$

The right-hand side of (3.6) is an integral. Integrating by parts we deduce

$$\langle u_1, u_2\rangle_2 = \int_I (t\dot{u}_1\dot{u}_2 + \mu^2 t^{-1} u_1 u_2 + t^3 m_2 u_1 u_2), \tag{3.7}$$

i.e., the scalar product is indeed positive definite because of the assumption $\mu > 0$. Let us define the norm

$$\|u\|_2^2 = \langle u, u\rangle_2 \qquad \forall\, u \in C^\infty_c(I) \tag{3.8}$$

and the Hilbert space $\mathcal{H}_2 = \mathcal{H}_2(I)$ as the closure of $C^\infty_c(I)$ with respect to the norm $\|\cdot\|_2$.

Proposition 3.3. *The functions $u \in \mathcal{H}_2$ have the properties*

$$u \in C^0([0,\infty)), \tag{3.9}$$

$$|u(t)| \le c\|u\|_2 \qquad \forall\, t \in I, \tag{3.10}$$

where $c = c(\mu, m_2, |\Lambda|)$,

$$\lim_{t\to 0} u(t) = 0 \tag{3.11}$$

and

$$|u(t)| \le c\|u\|_2 t^{-1} \qquad \forall\, t \in I, \tag{3.12}$$

where c is a different constant depending on μ, m_2 and $|\Lambda|$.

For a proof we refer to [8, Proposition 3.4.3, p. 82].

Theorem 3.4. *Let $u \in \mathcal{H}_2$ satisfy the equation*

$$A_1 u = -t^{-1}\frac{\partial}{\partial t}\big(t\frac{\partial u}{\partial t}\big) + t^{-2}\mu^2 u + t^2 m_2^2 u = \lambda t^q u, \tag{3.13}$$

where the constants μ, m_2 and λ are strictly positive and the exponent q satisfies

$$-2 < q < 2. \tag{3.14}$$

Since μ is especially important, let us emphasize that

$$\mu^2 = -\bar{\mu} = \frac{(m+k-1)^2}{4} + \gamma_0|\theta_0|^2 - \mu_0, \tag{3.15}$$

where γ_0 is a positive constant, $\theta_0 \in \mathbb{R}^k$ an arbitrary, but fixed, vector and $\mu_0 > 0$. Then, for every $\epsilon > 0$ there exists $0 < t_0 < 1$ and a positive constant c_1 such that u does not vanish in the interval $(0, t_0]$ and can be estimates by

$$|u(t)| \le c_1 t^{\mu_\epsilon} \qquad \forall\, t \in (0, t_0], \tag{3.16}$$

where $0 < \mu_\epsilon$ is defined by

$$\mu_\epsilon^2 = \mu^2 - \epsilon > 0. \tag{3.17}$$

Proof. Let us first prove that u does not vanish for small $t > 0$. Arguing by contradiction let $0 < t_0 < 1$ be a point where

$$u(t_0) = 0. \tag{3.18}$$

Multiplying the equation (3.13) by tu and integrating by parts over the interval $[0, t_0]$ we infer

$$\int_0^{t_0} \mu^2 t^{-1}|u|^2 \le \int_0^{t_0} \lambda t^{1+q}|u|^2 \tag{3.19}$$

and conclude further that t_0 cannot be arbitrarily close to 0.

Thus, let us assume u to be real valued and strictly positive in $(0, t_0]$ for some small t_0. To prove the inequality in (3.16), let us consider the equation

$$A_{1,\epsilon}\psi = \lambda\psi \qquad \text{in } (0,\infty) \tag{3.20}$$

requiring

$$\psi(0) = 0, \tag{3.21}$$

where the operator $A_{1,\epsilon}$ is defined by replacing μ by μ_ϵ in equation (3.13). One can easily verify that a solution $\psi = \psi(t)$ satisfying both equations is given by defining

$$\psi(t) = e^{\frac{1}{2}m_2 t^2} t^{\mu_\epsilon} M(a, b, m_2 t^2), \tag{3.22}$$

where

$$a = \frac{1}{2}(\mu_\epsilon + 1) - \frac{1}{4}\frac{\lambda}{m_2} \tag{3.23}$$

and

$$b = \mu_\epsilon + 1. \tag{3.24}$$

$M = M(a, b, z)$, $z \in \mathbb{C}$, is known as *Kummers's function* or as the entire *confluent hypergeometric function* which is a solution of *Kummer's equation*

$$zy'' + (b - z)y' - ay = 0 \tag{3.25}$$

and which can be expressed by the power series

$$ {}_1F_1(a, b, z) = M(a, b, z) = 1 + \frac{a}{b}z + \sum_{k=2}^{\infty} \frac{a(a+1)\cdots(a+k-1)z^k}{b(b+1)\cdots(b+k-1)k!} \tag{3.26}$$

which is absolutely convergent for any $z \in \mathbb{C}$ provided

$$b \notin \mathbb{Z}_{\le 0}, \tag{3.27}$$

which is certainly true in our case. For a detailed analysis of the solutions of Kummer's equation we refer to [11, Chapter 13.2, p. 322] or [9, p. 427].

Since u is a subsolution of the equation (3.20) in the interval $(0, t_0)$, i.e.,

$$A_{1,\epsilon} u \le \lambda u, \tag{3.28}$$

because

$$A_{1,\epsilon} u = \lambda t^q u - \epsilon t^{-2} u < 0 \tag{3.29}$$

if t_0 is small enough. Moreover, $\psi(t)$ is positive if t_0 is small, for $M(a, b, 0) = 1$, hence, there exists a constant c_2 such that

$$u(t_0) = c_2 \psi(t_0). \tag{3.30}$$

In order to prove (3.16) we multiply the inequality

$$A_{1,\epsilon}(u - c_2\psi) \le \lambda(u - c_2\psi) \tag{3.31}$$

by $t \max(u - c_2\psi, 0)$ and partially integrating the result in the interval $(0, t_0]$ yields

$$\int_0^{t_0} \mu_\epsilon^2 t^{-1} \max(u - c_2\psi, 0)^2 \le \int_0^{t_0} \lambda \max(u - c_2\psi, 0)^2 \tag{3.32}$$

from which we deduce

$$u(t) \le c_2 \psi(t) \qquad \forall\, t \in [0, t_0] \tag{3.33}$$

if t_0 is small, completing the proof of the theorem, in view of the definition of ψ in (3.22). □

Remark 3.5. The assumptions regarding the coefficients and the exponents of the ODE in the theorem above cover the cases we are confronted with after the quantization of the full Einstein equations, where $\theta_0 \in \mathbb{R}^k$ and $n \geq 3$ can be arbitrary and $q = 2 - \frac{2}{n}$, as well as in case of the Wheeler-DeWitt equation, where we have to choose $\theta_0 = 0$, n=3 and $q = -\frac{2}{3}$. To ensure that the right-hand side of equation (3.15) is positive in the latter case the dimension k of the target space of the scalar field map, which is $\mathbb{R}^k$, has to be sufficiently large.

We shall apply the estimate (3.16) to the function

$$\tilde{u} = t^{\frac{1}{2}} u, \tag{3.34}$$

which satisfies the differential equation

$$-\ddot{\tilde{u}} + t^{-2}\tilde{\mu}^2\tilde{u} + t^2 m_2^2 \tilde{u} = \lambda t^q \tilde{u}, \tag{3.35}$$

where

$$\tilde{\mu}^2 = \mu^2 - \frac{1}{4}, \tag{3.36}$$

as can be easily checked.

But before we shall prove that the eigenfunctions in equation (3.35) can be extended past the singularity as sufficiently smooth functions, let us verify that equation (3.35) is unitarily equivalent to equation (3.13), if we consider complex valued functions, otherwise there is an orthogonal equivalence. After that verification the countably many eigenfunctions $\tilde{u}_i$ with eigenvalues λ_i can be looked at as the temporal eigenfunctions of our model of quantum gravity which can be extended past the singularity. We shall also prove that equation (3.35) can be defined as a classical equation for $(\tilde{u}_i, \lambda_i)$ valid in $\mathbb{R}$ provided t^q is replaced by $|t|^q$ and $\tilde{u}_i$ is extended by reflection either even or odd.

Definition 3.6. For functions $u \in C_c^\infty(I)$ define the operators

$$A_r u = -t^{-r}\frac{\partial}{\partial t}\big(t^r \frac{\partial u}{\partial t}\big) + t^{-2}\mu^2 u + t^2 m_2 u \tag{3.37}$$

and the scalar product

$$\langle u_1, u_2\rangle_2 = \langle A_r u_1, u_2\rangle_r \qquad \forall\, u_1, u_2 \in C_c^\infty(I). \tag{3.38}$$

The right-hand side of (3.38) is an integral. Integrating by parts we deduce

$$\langle u_1, u_2\rangle_2 = \int_I (t^r \dot{u}_1\dot{u}_2 + \mu^2 t^{r-2} u_1 u_2 + t^{r+2} m_2 u_1 u_2). \tag{3.39}$$

Let us define the norm

$$\|u\|_2^2 = \langle u, u\rangle_2 \qquad \forall\, u \in C_c^\infty(I) \tag{3.40}$$

and the Hilbert space $\mathcal{H}_2 = \mathcal{H}_2(I)$ as the closure of $C_c^\infty(I)$ with respect to the norm $\|\cdot\|_2$.

Define the operator A_0 in $C_c^\infty(I)$ by

$$A_0 = -\ddot{\tilde{u}} + t^{-2}\tilde{\mu}^2\tilde{u} + t^2 m_2\tilde{u}, \tag{3.41}$$

where

$$\tilde{\mu}^2 = \mu^2 + \frac{r^2}{4} - \frac{r}{2} \tag{3.42}$$

is supposed to be strictly positive and let $\tilde{\mathcal{H}}_2$ be the completion with respect to the corresponding norm

$$|||\tilde{u}|||^2 = \int_0^\infty (|\dot{\tilde{u}}|^2 + t^{-2}\tilde{\mu}^2\tilde{u}^2 + t^2 m_2\tilde{u}^2) = \langle A_0\tilde{u}, \tilde{u}\rangle \equiv \langle\langle \tilde{u}, \tilde{u}\rangle\rangle_2 \tag{3.43}$$

Proposition 3.7. *The functions $\tilde{u} \in \tilde{\mathcal{H}}_2$ have the properties*

$$\tilde{u} \in C^0([0,\infty)), \tag{3.44}$$

$$|\tilde{u}(t)| \le c|||\tilde{u}|||_2 \qquad \forall\, t \in I, \tag{3.45}$$

where $c = c(\bar{\mu}, m_2)$,

$$|\tilde{u}(t)| \le c|||\tilde{u}|||_2 t^{\frac{1}{2}} \qquad \forall\, t \in I, \tag{3.46}$$

as well as

$$|\tilde{u}(t)| \le c|||\tilde{u}|||_2 t^{-\frac{1}{2}} \qquad \forall\, t \in I, \tag{3.47}$$

where c is a different constant depending on $\tilde{\mu}, m_2$.

Proof. Let us first assume $\tilde{u} \in C_c^\infty(I)$ and let $\delta > 0$, then

$$\tilde{u}^2(\delta) = 2\int_0^\delta \dot{\tilde{u}}\tilde{u} \le \int_0^\delta |\dot{\tilde{u}}|^2 + \int_0^\delta |\tilde{u}|^2. \tag{3.48}$$

This estimate is also valid for any $\tilde{u} \in \tilde{\mathcal{H}}_2$ by approximation which in turn implies the relations (3.45).

Next let us slightly modify the previous argument to obtain

$$\tilde{u}^2(\delta) = 2\int_0^\delta \dot{\tilde{u}}\tilde{u} \le 2\Big(\int_0^\delta \dot{\tilde{u}}^2\Big)^{\frac{1}{2}}\Big(\int_0^\delta t^{-2}t^2\tilde{u}^2\Big)^{\frac{1}{2}} \le c|||\tilde{u}|||_2^2\,\delta \tag{3.49}$$

from which we infer (3.46) and also (3.44) since u is continuous in I.

It remains to prove (3.47). Let $\tilde{u} \in \tilde{\mathcal{H}}_2$ and define $\tilde{v} = \tilde{v}(\tau)$ by

$$\tilde{v}(\tau) = \tilde{u}(\tau^{-1}), \tag{3.50}$$

where $\tau = t^{-1}$ for all $t > 0$. Applying simple calculus arguments we then obtain

$$\int_0^\infty \{\tau^2|\tilde{v}'|^2 + \tau^2\tilde{\mu}^2|\tilde{v}|^2 + \tau^{-4}m_2|\tilde{v}|^2\}d\tau = |||\tilde{u}|||_2^2 \tag{3.51}$$

as well as

$$\int_0^\infty \{\tau^2|\tilde{v}'|^2 + \tau^2\tilde{\mu}^2|\tilde{v}|^2\}d\tau = \int_0^\infty \{|\dot{\tilde{u}}|^2 + t^{-2}\tilde{\mu}^2|\tilde{u}|^2\}dt. \tag{3.52}$$

Moreover, first assuming, as before, that $\tilde{u}$ and hence $\tilde{v}$ are test functions we argue as in (3.49) that for any $\delta > 0$

$$\begin{aligned}\tilde{v}^2(\delta) = 2\int_0^\delta \tilde{v}'\tilde{v} &\le 2\Big(\int_0^\delta \tau^2|\tilde{v}'|^2\Big)^{\frac12}\Big(\int_0^\delta \tau^{-2}|\tilde{v}|^2\Big)^{\frac12}\\ &\le 2\Big(\int_0^\delta \tau^2|\tilde{v}'|^2\Big)^{\frac12}\Big(\int_0^\delta \tau^{-4}|\tilde{v}|^2\Big)^{\frac12}\delta\\ &\le c|||\tilde{u}|||_2^2\,\delta,\end{aligned} \tag{3.53}$$

where we used (3.51) for the last inequality and where $c = c(\tilde{\mu}, m_2)$. Setting $\delta = t^{-1}$ for arbitrary $t > 0$ we have proved the estimate (3.47) for test functions and hence for arbitrary $\tilde{u} \in \tilde{\mathcal{H}}_2$. ☐

Lemma 3.8. *Assuming the definitions in Definition 3.6, then the map*

$$\begin{aligned}\varphi : \mathcal{H}_2 &\to \tilde{\mathcal{H}}_2,\\ u &\to \tilde{u} = t^{\frac{r}{2}}u\end{aligned} \tag{3.54}$$

is orthogonal if the functions are supposed to be real valued and unitary if complex functions are considered and the scalar products are suitably modified, i.e,

$$\langle A_r u_1, u_2\rangle_r = \langle A_0\tilde{u}_1, \tilde{u}_2\rangle \qquad \forall\, u_i \in \mathcal{H}_2,\ i = 1, 2, \tag{3.55}$$

and

$$\begin{aligned}A_r &= \varphi^{-1}\circ A_0 \circ \varphi,\\ A_0 &= \varphi \circ A_r \circ \varphi^{-1},\end{aligned} \tag{3.56}$$

i.e., A_r and A_0 are unitarily equivalent.

Proof. For the prove of (3.55) we may assume that the functions are real valued. The relation is then easily verified by applying elementary calculus:

$$t^r\dot{u}_1\dot{u}_2 = \dot{\tilde{u}}_1\dot{\tilde{u}}_2 + \frac{r^2}{4}t^{-2}\tilde{u}_1\tilde{u}_2 - \frac{r}{2}t^{-1}(\tilde{u}_1\tilde{u}_2)' \tag{3.57}$$

from which we deduce by applying partial integration

$$\begin{aligned}&\int_0^\infty \{t^r\dot{u}_1\dot{u}_2 + \mu^2 t^{r-2}u_1u_2 + m_2t^{r+2}u_1u_2\} =\\ &\int_0^\infty \{\dot{\tilde{u}}_1\dot{\tilde{u}}_2 + (\mu^2 + \frac{r^2}{4} - \frac{r}{2})t^{-2}\tilde{u}_1\tilde{u}_2 + m_2t^2\tilde{u}_1\tilde{u}_2\}.\end{aligned} \tag{3.58}$$

Moreover, a straightforward calculation reveals that for test functions u

$$A_0\tilde{u} = t^{\frac{r}{2}}A_r u \tag{3.59}$$

proving (3.56). ☐

The equations (3.35) resp.

$$A_r = \lambda t^q u \tag{3.60}$$

can be looked at as eigenvalue equations which can be expressed abstractly in the form: $u \in \mathcal{H}_2$ satisfies

$$B(u,v) \equiv \langle A_r u, v\rangle_r = \lambda K(u,v) \qquad \forall\, v \in \mathcal{H}_2, \tag{3.61}$$

where

$$K(u,v) = \int_0^\infty t^{r+q} uv \tag{3.62}$$

and where we only consider real valued functions for simplicity. Since $r \in \mathbb{R}$ is arbitrary the case $r = 0$ is also covered.

Theorem 3.9. *The eigenvalue problem* (3.61) *is orthogonally (unitarily) equivalent to the corresponding eigenvalue problem: $\tilde u \in \tilde{\mathcal{H}}_2$ satisfies*

$$\tilde B(\tilde u, \tilde v) \equiv \langle A_0 \tilde u, \tilde v\rangle = \lambda \tilde K(\tilde u, \tilde v) \qquad \forall\, \tilde v \in \tilde{\mathcal{H}}_2, \tag{3.63}$$

where

$$\tilde K(\tilde u, \tilde v) = \int_0^\infty t^q \tilde u \tilde v. \tag{3.64}$$

Hence, the respective eigenvalues are identical.

Proof. Let φ be the unitary map in Lemma 3.8. In view of (3.55) we conclude

$$B(u,v) = \tilde B(\varphi(u), \varphi(v)) \tag{3.65}$$

and also

$$K(u,v) = \tilde K(\varphi(u), \varphi(v)) \tag{3.66}$$

completing the proof. □

If $q \in \mathbb{R}$ satisfies the estimates

$$-2 < q < 2 \tag{3.67}$$

then the quadratic form

$$\tilde K(v) = \tilde K(v,v) \tag{3.68}$$

is *compact* with respect to the quadratic form

$$\tilde B(\tilde v) = \tilde B(\tilde v, \tilde v), \tag{3.69}$$

i.e., if

$$\tilde B(\tilde v_i, \tilde v) \to \tilde B(\tilde v_0), \tilde v) \qquad \forall\, \tilde v \in \tilde{\mathcal{H}}_2 \tag{3.70}$$

then

$$\tilde K(\tilde v_i - \tilde v_0) \to 0. \tag{3.71}$$

The proof is well-known and fairly simple: In compact subintervals of $(0,\infty)$ the compactness follows from the Sobolev embedding theorems and near the endpoints of the interval $t = 0$ and $t = \infty$ the compactness can be deduced from the finiteness of

$$\int_0^\infty (t^{-2} + t^2)|\tilde v_i - \tilde v_0|^2 \le \text{const} \qquad \forall\, i \in \mathbb{N}. \tag{3.72}$$

The latter estimate is due to the definition of the scalar product $\tilde B$ and the uniform boundedness principle which says that any weakly bounded sequence in a Banach space is uniformly bounded.

If these conditions are satisfied then the following theorem is well-known:

Theorem 3.10. *The eigenvalue problem (3.63) has countably many solutions $(\lambda_i, \tilde u_i)$, $\tilde u_i \in \tilde{\mathcal{H}}_2$, with the properties*

$$\lambda_i < \lambda_{i+1} \qquad \forall\, i \in \mathbb{N}, \tag{3.73}$$

$$\lim_i \lambda_i = \infty, \tag{3.74}$$

$$\tilde K(\tilde u_i, \tilde u_j) = \delta_{ij}. \tag{3.75}$$

The pairs $(\lambda_i, \tilde u_i)$ are recursively defined by the variational problems

$$\lambda_0 = \tilde B(\tilde u_0) = \inf \left\{ \frac{\tilde B(u)}{\tilde K(u)} : 0 \ne u \in \tilde{\mathcal{H}}_2 \right\} \tag{3.76}$$

and for $i > 0$

$$\lambda_i = \tilde B(\tilde u_i) = \inf \left\{ \frac{\tilde B(u)}{\tilde K(u)} : 0 \ne u \in \tilde{\mathcal{H}}_2,\ \tilde K(u, u_j) = 0,\ 0 \le j \le i-1 \right\}. \tag{3.77}$$

The $(\tilde u_i)$ form a Hilbert space basis in $\tilde{\mathcal{H}}_2$ and in $L^2(I, q)$, the eigenvalues are strictly positive and the eigenspaces are one dimensional.

Proof. This theorem is well-known and goes back to the book of Courant-Hilbert [2], though in a general separable Hilbert space the eigenvalues are not all positive and the eigenspaces are only finite dimensional. For a proof in the general case we refer to [3, Theorem 1.6.3, p. 37].

The positivity of the eigenvalues in the above theorem is obvious and the fact that the eigenspaces are one dimensional is proved by contradiction. Thus, suppose there exist an eigenvalue $\lambda = \lambda_i$ and two corresponding linearly independent eigenfunctions $\tilde u_1, \tilde u_2 \in \tilde{\mathcal{H}}_2$. Then, for any $t_0 > 0$ there would exist an eigenfunction $u \in \tilde{\mathcal{H}}_2$ with eigenvalue λ satisfying $u(t_0) = 0$ and the equation (3.63). Multiplying this equation by u and integrating the result in the interval $(0, t_0)$ with respect to the measure dt we obtain

$$\int_0^{t_0} \tilde\mu^2 t^{-2} u^2 \le t_0^{2+q} \int_0^{t_0} \lambda t^{-2} u^2, \tag{3.78}$$

where we used

$$1 \le \frac{t_0}{t}, \qquad \forall\, t \in (0, t_0), \tag{3.79}$$

and

$$2 + q > 0, \tag{3.80}$$

in view of (3.14), yielding a contradiction if t_0 is sufficiently small. □

Remark 3.11. The previous results are a also valid if instead of the coefficient

$$m_2 t^2 \tag{3.81}$$

we consider the actual coefficient

$$m_2 |\Lambda| t^2, \tag{3.82}$$

where in our case $\Lambda < 0$. The eigenvalues λ_i then depend on Λ.

In [8, Lemma 9.4.8, p. 240] we proved the following lemma, which we include here together with an appropriately modified proof for the convenience of the reader.

Lemma 3.12. *Let λ_i be the temporal eigenvalues depending on Λ and let $\bar\lambda_i$ be the corresponding eigenvalues for*

$$|\Lambda| = 1, \tag{3.83}$$

then

$$\lambda_i = \bar\lambda_i |\Lambda|^{\frac{2+q}{4}}. \tag{3.84}$$

Proof. Let $\tilde B$ and $\tilde K$ be the quadratic forms defined by

$$\tilde B(u) = \int_0^\infty \{|\dot u|^2 + t^{-2}|\tilde\mu|^2|u|^2 + t^2 m_2 |\Lambda||u|^2\} \tag{3.85}$$

and

$$\tilde K(u) = m_3 \int_0^\infty t^q |u|^2 \tag{3.86}$$

and let $\tilde B_1(u)$ the quadratic form by choosing $|\Lambda| = 1$ in $\tilde B$. Then we have

$$\frac{\tilde B(u)}{\tilde K(u)} = |\Lambda|^{\frac{2+q}{4}} \frac{\tilde B_1(u)}{\tilde K(u)} \qquad \forall\, 0 \ne u \in C_c^\infty(\mathbb{R}_+). \tag{3.87}$$

To prove (3.87) we introduce a new integration variable τ on the left-hand side

$$t = \mu\tau, \qquad \mu > 0, \tag{3.88}$$

to conclude

$$\frac{\tilde B(u)}{\tilde K(u)} = \mu^{-(2+q)} \frac{\tilde B_1(u)}{\tilde K(u)} \qquad \forall\, 0 \ne u \in C_c^\infty(\mathbb{R}_+). \tag{3.89}$$

provided

$$\mu = |\Lambda|^{-\frac14}. \tag{3.90}$$

The relation (3.87) immediately implies (3.84), in view of Theorem 3.10. □

Remark 3.13. Let $(\tilde{u}_i, \lambda_i)$ be the previous eigenfunctions and eigenvalues of the operator

$$-\ddot{\tilde{u}} + t^{-2}\tilde{\mu}^2\tilde{u} + t^2 m_2^2|\Lambda|\tilde{u} \tag{3.91}$$

with respect to the quadratic form

$$\tilde{K}(\tilde{u}) = \int_0^\infty t^q|\tilde{u}|^2, \tag{3.92}$$

define

$$\varphi_0(t) = t^q \tag{3.93}$$

and let H_0 be the operator

$$\varphi_0^{-1}(-\ddot{\tilde{u}} + t^{-2}\tilde{\mu}^2\tilde{u} + t^2 m_2^2|\Lambda|\tilde{u}) \tag{3.94}$$

defined in the dense subspace of the Hilbert space $\mathcal{H} = L^2(I, \varphi_0 dt)$ generated by the eigenfunctions $(\tilde{u}_i)$, then H_0 is essentially self-adjoint and its closure, which we denote by the same symbol, is self-adjoint; for a proof see the remarks following [8, Definition 3.4.14, p.91].

In the next section we shall prove that for any $\beta > 0$

$$e^{-\beta H_0} \tag{3.95}$$

is of trace class in $\mathcal{H}$, i.e.,

$$\operatorname{tr}(e^{-\beta H_0}) = \sum_{i=0}^\infty e^{-\beta\lambda_i} < \infty. \tag{3.96}$$

Because we consider arbitrary q satisfying

$$-2 < q < 2 \tag{3.97}$$

and not only the special values

$$q = 2 - \frac{2}{n} \quad \vee \quad q = -\frac{2}{3} \tag{3.98}$$

we cannot refer to a previous result and an extra proof is necessary.

After having established that $\tilde{u}$ is unitarily equivalent to the solution u of (3.13) which in turn is unitarily equivalent to the solution w of equation (1.31) on page 5 resp. (2.28) on page 12, cf. [8, Lemma 3.4.10, p. 89], we shall consider the equation (3.35) and its solution $\tilde{u}$, defined in (3.34), to be the temporal eigenfunction equation which we shall extend past the singularity. In view of the estimate (3.16), where μ_ϵ is defined in (3.17) we infer, by using the fact that we may assume u to be positive in $(0, t_0]$,

$$0 < \tilde{u} \le c_1 t^{\mu_\epsilon + \frac{1}{2}} \qquad \forall\, t \in (0, t_0], \tag{3.99}$$

where $\epsilon > 0$ is as small as we like but fixed. The constant c_1 depends on ϵ and will tend to infinity if ϵ tends to zero. However, we are able to conclude

Lemma 3.14. *Let* $1 \le m_0 \in \mathbb{N}$ *and assume*

$$\mu + \frac{1}{2} > m_0, \tag{3.100}$$

then there exists $\epsilon > 0$ *and positive constants* c_1, t_0 *such that*

$$0 < \tilde{u} \le c_1 t^{m_0+\epsilon} \qquad \forall\, t \in I = (0, t_0]. \tag{3.101}$$

The proof is obvious.

Lemma 3.15. *Let the assumption* (3.100) *be satisfied for* $m_0 = 1$*, then*

$$\tilde{u} \in C^1[0, t_0] \quad \wedge \quad \dot{\tilde{u}}(0) = 0. \tag{3.102}$$

Moreover, $\tilde{u}$ *is strictly convex in* $[0, t_0]$ *if* t_0 *is small enough. Extending* $\tilde{u}$ *to* $[-t_0, 0)$ *by defining*

$$\tilde{u}(t) = \begin{cases} \tilde{u}(t), & t \ge 0, \\ \tilde{u}(-t), & t < 0, \end{cases} \tag{3.103}$$

then the extended function is of class C^1 *in* $[-t_0, t_0]$*, strictly convex and*

$$\ddot{\tilde{u}} > 0 \tag{3.104}$$

in the distributional sense, i.e.,

$$\langle \tilde{u}, \ddot{\eta} \rangle > 0 \qquad \forall\, 0 \le \eta \in C_c^\infty(-t_0, t_0), \tag{3.105}$$

which do not vanish identically.

Proof. From the equation (3.35) we deduce

$$\ddot{\tilde{u}}(t) > 0 \qquad \forall\, t \in (0, t_0), \tag{3.106}$$

if t_0 is small enough, hence $\tilde{u}$ is strictly convex in the interval. Since $\tilde{u} > 0$ and and $\tilde{u}(0) = 0$, we infer

$$\dot{\tilde{u}}(t) > 0 \qquad \forall\, t \in I, \tag{3.107}$$

because $\dot{\tilde{u}}$ is also monotone increasing. Hence, we conclude

$$0 \le c = \lim_{t \to 0} \dot{\tilde{u}}(t) \tag{3.108}$$

exists. If $c > 0$ we would obtain a contradiction in view of (3.101), i.e., the right derivative of $\tilde{u}$ satisfies

$$\dot{\tilde{u}}(0) = \lim_{t \to 0} \frac{\tilde{u}(t)}{t} = 0 = \lim_{t \to 0} \dot{\tilde{u}}(t), \tag{3.109}$$

hence, we have proved (3.102).

Finally, (3.104) is valid for any $0 \ne t \in (-t_0, t_0)$ and the relation (3.105) follows by partial integration over the open subintervals $\{t \ne 0\}$ by using

$$\tilde{u}(0) = 0 = \dot{\tilde{u}}(0). \tag{3.110}$$

□

Lemma 3.16. *Let the assumption (3.100) be satisfied for $m_0 = 2$, then*

$$\tilde{u} \in C^2([0, t_0]), \tag{3.111}$$

$$\lim_{t\to 0} \frac{\tilde{u}(t)}{t^2} = 0 \tag{3.112}$$

and

$$\ddot{\tilde{u}}(0) = 0 = \lim_{t\to 0} \ddot{\tilde{u}}(t) \quad \wedge \quad \lim_{t\to 0} \frac{\dot{\tilde{u}}(t)}{t} = 0. \tag{3.113}$$

Moreover, these properties are also valid for the extended function.

Proof. The equation (3.112) is due to (3.101), while the first relation in (3.113) immediate follows from (3.112) and the equation satisfied by $\tilde{u}$.

To prove the second equation in (3.113) we apply De L'Hospital's rule and use the first equation. Finally, it is obvious that these properties are also valid for the extended function. □

We are now able to prove by induction

Theorem 3.17. *Let the assumption (3.100) be satisfied for arbitrary $2 \le m_0 \in \mathbb{N}$, then*

$$\tilde{u} \in C^{m_0}([0, t_0]) \quad \wedge \quad \tilde{u}^{(m_0)}(0) = 0 = \lim_{t\to 0} \tilde{u}^{(m_0)}(t) \tag{3.114}$$

as well as

$$\lim_{t\to 0} \frac{\tilde{u}^{(k)}(t)}{t^{m_0-k}} = 0 \qquad \forall\, 1 \le k \le m_0,\ k \in \mathbb{N}, \tag{3.115}$$

where $\tilde{u}^{(k)}$ denotes the k-th derivative of $\tilde{u}$. These properties are also valid for the extended function.

Proof. The claims in (3.114) are certainly correct provided the relations in (3.115) are valid. Hence, it suffices to prove the relations in (3.115) per induction with respect to k. Let us first consider the case $k = 1$. Applying De L'Hospital's rule we deduce

$$\lim_{t\to 0} \frac{\dot{\tilde{u}}(t)}{t^{m_0-1}} = (m_0-1)^{-1} \lim_{t\to 0} \frac{\ddot{\tilde{u}}(t)}{t^{m_0-2}} = (m_0-1)^{-1} \lim_{t\to 0} \frac{\tilde{u}(t)}{t^{m_0}} = 0, \tag{3.116}$$

where we used for the second equality the equation satisfied by $\tilde{u}$ and for the last the estimate (3.101). The last two arguments also reveal that the claim in (3.115) is true for $k = 2$.

Thus, let us assume that the limit relations in (3.115) are already valid for $1 \le k \le p < m_0$, $p \ge 2$, and let us prove that then they are also satisfied for $k = p + 1$. Let us recall that $\tilde{u}$ is a solution of the equation (3.35) which we can write in the form

$$\ddot{\tilde{u}} = \tilde{\mu}^2 t^{-2} \tilde{u} + (m_2^2 t^2 - \lambda t^q)\tilde{u}. \tag{3.117}$$

Differentiating both sides with respect to D^{p-1}, where D denotes differentiation with respect to t, we deduce, by applying the product rule,

$$\tilde{u}^{(p+1)} = \tilde{\mu}^2 \sum_{k=0}^{p-1} c_{p,k} t^{-2-k} \tilde{u}^{(p-1-k)} + R_1 + R_2, \tag{3.118}$$

where the additional terms R_1, R_2 have a similar structure as the detailed sum, but the exponents of t are less critical for small $t > 0$ than in the first sum. The arguments we shall use in the case of the first sum will also apply in case of the additional terms and will therefore be omitted.

Next, we have to prove

$$\lim_{t\to 0} \frac{\tilde{u}^{(p+1)}}{t^{m_0-(p+1)}} = 0. \tag{3.119}$$

Indeed, we infer

$$\sum_{k=0}^{p-1} c_{p,k} \frac{\tilde{u}^{(p-1-k)}}{t^{2+k+m_0-(p+1)}} = \sum_{k=0}^{p-1} c_{p,k} \frac{\tilde{u}^{(p-1-k)}}{t^{m_0-(p-1-k)}} \tag{3.120}$$

and the right-hand side converges to zero if t tends to zero, in view of the induction assumption. Hence, the relation (3.119) is proved completing the proof of the theorem. □

As a corollary we obtain

Corollary 3.18. *If the assumption of the preceding theorem is satisfied then the extended solution $\tilde{u}$ also satisfies the extended equation*

$$-\ddot{\tilde{u}} + t^{-2}\tilde{\mu}^2\tilde{u} + t^2 m_2^2 \tilde{u} = \lambda |t|^q \tilde{u} \tag{3.121}$$

in $\mathbb{R}$, where we have to replace t^q by $|t|^q$ for obvious reasons. Let us emphasize that the lower order coefficients of the ODE exhibit a singularity in $t = 0$ but that both sides of the equation are continuous in the interval $(-\infty, \infty)$ and vanish in $t = 0$.

4. Trace class estimates for $e^{-\beta H_0}$

We consider the operator H_0 in (3.94) on page 23 which is essentially self-adjoint in

$$\mathcal{H} = L^2(\mathbb{R}_+, d\mu), \tag{4.1}$$

where

$$d\mu = \varphi_0 dt \tag{4.2}$$

with

$$\varphi_0(t) = t^q, \tag{4.3}$$

where q satisfies the relation (3.97) and we shall also use the same symbol for its closure, i.e., we shall assume that H_0 is self-adjoint in $\mathcal{H}$ with eigenvectors $u_i \in \tilde{\mathcal{H}}_2$ and with eigenvalues λ_i satisfying the statements in Theorem 3.10

on page 21. However, now we denote the eigenvectors by u_i to improve the readability.

Remark 4.1. The norm

$$\langle H_0 u, u\rangle^{\frac{1}{2}} \tag{4.4}$$

is equivalent to the norm $|||u|||_2$ in $\tilde{\mathcal{H}}_2$, since $|\Lambda| > 0$.

Let us also assume that all Hilbert spaces are complex vector spaces with a positive definite sesquilinear form (hermitian scalar product).

We shall now prove that

$$e^{-\beta H_0}, \qquad \beta > 0, \tag{4.5}$$

is of trace class in $\mathcal{H}$. The proof is essentially the proof given in [8, Chapter 3.5] with the necessary modifications due to the different exponent in $\varphi_0(t)$.

First, we need two lemmata:

Lemma 4.2. *The embedding*

$$j : \tilde{\mathcal{H}}_2 \hookrightarrow \mathcal{H}_0 = L^2(\mathbb{R}_+, d\tilde{\mu}), \tag{4.6}$$

where

$$d\tilde{\mu} = (1+t)^{-2} dt, \tag{4.7}$$

is Hilbert-Schmidt, i.e., for any ONB (e_i) *in* $\tilde{\mathcal{H}}_2$ *the sum*

$$\sum_{i=0}^{\infty} \|j(e_i)\|_0^2 < \infty \tag{4.8}$$

is finite, where $\|\cdot\|_0$ *is the norm in* $\mathcal{H}_0$*. The square root of the left-hand side of* (4.8) *is known as the Hilbert-Schmidt norm* $|j|$ *of* j *and it is independent of the ONB, cf.* [10, Lemma 1, p. 158].

Proof. Let $w \in \tilde{\mathcal{H}}_2$, then, assuming w is real valued,

$$\begin{aligned} |w(t)|^2 = 2\int_0^t \dot{w} w &\le \int_o^\infty |\dot{w}|^2 + \int_0^\infty |w|^2 \\ &\le c|||w|||_2^2 \end{aligned} \tag{4.9}$$

for all $t > 0$, where $|||\cdot|||_2$ is the norm in $\tilde{\mathcal{H}}_2$. To derive the last inequality in (4.9) we used (3.43) on page 18. The estimate

$$|w(t)| \le c|||w|||_2 \qquad \forall\, t > 0 \tag{4.10}$$

is of course also valid for complex valued functions from which infer that, for any $t > 0$, the linear form

$$w \to w(t), \qquad w \in \tilde{\mathcal{H}}_2, \tag{4.11}$$

is continuous, hence it can be expressed as

$$w(t) = \langle\langle \varphi_t, w\rangle\rangle_2, \tag{4.12}$$

where

$$\varphi_t \in \tilde{\mathcal{H}}_2 \tag{4.13}$$

and

$$|||\varphi_t|||_2 \le c, \tag{4.14}$$

in view of (4.10). Now, let

$$e_i \in \tilde{\mathcal{H}}_2 \tag{4.15}$$

be an ONB, then

$$\sum_{i=0}^{\infty} |e_i(t)|^2 = \sum_{i=0}^{\infty} |\langle\langle \varphi_t, e_i\rangle\rangle_2|^2 = |||\varphi_t|||_2^2 \le c^2. \tag{4.16}$$

Integrating this inequality over $\mathbb{R}_+$ with respect to $d\tilde{\mu}$ we infer

$$\sum_{i=0}^{\infty} \int_0^\infty |e_i(t)|^2 d\tilde{\mu} \le c^2 \tag{4.17}$$

completing the proof of the lemma. □

Lemma 4.3. *Let u_i be the eigenfunctions of H_0, then there exist positive constants c and γ such that*

$$|||u_i|||_2 \le c|1+\lambda_i|^\gamma \|u_i\|_0 \qquad \forall\, i \in \mathbb{N}, \tag{4.18}$$

where $\|\cdot\|_0$ is the norm in $\mathcal{H}_0$.

Proof. We have

$$\langle H_0 u_i, u_i\rangle = \lambda_i \langle u_i, u_i\rangle \tag{4.19}$$

and hence, in view of Remark 4.1,

$$\begin{aligned} |||u_i|||_2^2 &\le c_1\lambda_i \int_0^\infty \varphi_0(t)|u_i|^2 \\ &\le c_1\lambda_i \Big\{ \int_0^1 \varphi_0(t)|u_i|^2 + c_2\int_1^\infty t^{2-\frac{2}{l_0}}|u_i|^2 \Big\}, \end{aligned} \tag{4.20}$$

where l_0 is very large such that

$$q < 2 - \frac{2}{l_0}. \tag{4.21}$$

To estimate the second integral in the braces let us define $p=2$ and such that

$$t^q \le t^{2-\frac{2}{l_0}} = t^{p-\frac{p}{l_0}} \qquad \forall\, t \ge 1. \tag{4.22}$$

Then, choosing small positive constants δ and ϵ, we apply Young's inequality, with

$$q_0 = \frac{p}{p-p\delta} = \frac{1}{1-\delta} \tag{4.23}$$

and

$$q_0' = \delta^{-1} \tag{4.24}$$

to estimate the integral from above by

$$\begin{aligned} &\frac{1}{q_0}\epsilon^{q_0}\int_1^\infty \{t^{p-\frac{p}{l_0}}(1+t)^{\frac{p}{l_0}-p\delta}\}^{q_0}|u_i|^2 \\ &\qquad\qquad + \frac{1}{q_0'}\epsilon^{-q_0'}\int_1^\infty (1+t)^{-(\frac{p}{l_0}-p\delta)q_0'}|u_i|^2. \end{aligned} \tag{4.25}$$

Choosing now δ so small such that

$$(\frac{p}{l_0}-p\delta)\delta^{-1} > 2 \tag{4.26}$$

the preceding integrals can be estimated from above by

$$\frac{1}{q_0}\epsilon^{q_0}\int_1^\infty (1+t)^p|u_i|^2 + \frac{1}{q_0'}\epsilon^{-q_0'}\int_0^\infty (1+t)^{-2}|u_i|^2 \tag{4.27}$$

which in turn can be estimated by

$$\frac{1}{q_0}\epsilon^{q_0}c|\!|\!|u_i|\!|\!|_2^2 + \frac{1}{q_0'}\epsilon^{-q_0'}\|u_i\|_0^2, \tag{4.28}$$

in view of (3.43) on page 18.

Since $-2 < q$ there exists ϵ_0 such that

$$-(2-2\epsilon_0) < q, \tag{4.29}$$

hence, using again Young's inequality, the first integral in the braces on the right-hand side of (4.20) can be estimated by

$$\begin{aligned} \int_0^1 \varphi_0(t)|u_i|^2 &\le c\int_0^1 t^{-(2-2\epsilon_0)}|u_i|^2 \le c(1-\epsilon_0)\epsilon^{\frac{1}{1-\epsilon_0}}\int_0^1 t^{-2}|u_i|^2 \\ &\qquad + c\epsilon_0\epsilon^{-\frac{1}{\epsilon_0}}\int_0^\infty (1+t)^{-2}|u_i|^2 \\ &\le \tilde{c}(1-\epsilon_0)\epsilon^{\frac{1}{1-\epsilon_0}}|\!|\!|u_i|\!|\!|_2^2 + c\epsilon_0\epsilon^{-\frac{1}{\epsilon_0}}\|u_i\|_0^2. \end{aligned} \tag{4.30}$$

Choosing now ϵ, γ and c appropriately the result follows. □

We are now ready to prove:

Theorem 4.4. *Let $\beta > 0$, then the operator*

$$e^{-\beta H_0} \tag{4.31}$$

is of trace class in $\mathcal{H}$, i.e.,

$$\operatorname{tr}(e^{-\beta H_0}) = \sum_{i=0}^\infty e^{-\beta\lambda_i} = c(\beta) < \infty. \tag{4.32}$$

Proof. In view of Lemma 4.2 the embedding

$$j : \tilde{\mathcal{H}}_2 \hookrightarrow \mathcal{H}_0 \tag{4.33}$$

is Hilbert-Schmidt. Let

$$u_i \in \mathcal{H} \tag{4.34}$$

be an ONB of eigenfunctions, then

$$\begin{aligned} e^{-\beta\lambda_i} &= e^{-\beta\lambda_i}\|u_i\|^2 \le e^{-\beta\lambda_i} c\lambda_i^{-1} |||u_i|||_2^2 \\ &\le e^{-\beta\lambda_i}\lambda_i^{-1} c|\lambda_i + 1|^{2\gamma} \|u_i\|_0^2, \end{aligned} \tag{4.35}$$

in view of (4.19) and (4.18), but

$$\|u_i\|_0^2 = |||u_i|||_2^2 \, \|\tilde{u}_i\|_0^2 \le c\lambda_i \|\tilde{u}_i\|_0^2, \tag{4.36}$$

where

$$\tilde{u}_i = u_i |||u_i|||_2^{-1} \tag{4.37}$$

is an ONB in $\mathcal{H}_2$, yielding

$$\sum_{i=0}^{\infty} e^{-\beta\lambda_i} \le c_\beta \sum_{i=0}^{\infty} \|\tilde{u}_i\|_0^2 < \infty, \tag{4.38}$$

since j is Hilbert-Schmidt. Here we used Remark 4.1, since the scalar product in $\tilde{\mathcal{H}}_2$ has to be defined by

$$\langle H_0 u, v\rangle \tag{4.39}$$

in order to deduce that the eigenfunctions are also mutually orthogonal in $\tilde{\mathcal{H}}_2$, and also $\lambda_0 > 0$. □

Remark 4.5. This result enables us to apply quantum statistics to our model of quantum gravity and to define a partition function Z, a density operator ρ and the von Neumann entropy S in a corresponding Fock space. For details we refer to [8, Chapter 9.5].

References

[1] R. Arnowitt, S. Deser, and C. W. Misner, *The dynamics of general relativity*, Gravitation: an introduction to current research (Louis Witten, ed.), John Wiley, New York, 1962, pp. 227–265.

[2] R. Courant and D. Hilbert, *Methoden der mathematischen Physik. I*, Springer-Verlag, Berlin, 1968, doi:10.1007/978-3-642-47436-1, Dritte Auflage, Heidelberger Taschenbücher, Band 30.

[3] Claus Gerhardt, *Partial differential equations II*, Lecture Notes, University of Heidelberg, 2013, pdf file.

[4] ———, *The quantization of gravity in globally hyperbolic spacetimes*, Adv. Theor. Math. Phys. **17** (2013), no. 6, 1357–1391, arXiv:1205.1427, doi:10.4310/ATMP.2013.v17.n6.a5.

[5] ———, *The quantization of gravity: Quantization of the Hamilton equations*, Universe **7** (2021), no. 4, 91, doi:10.3390/universe7040091.

[6] ———, *A unified quantization of gravity and other fundamental forces of nature*, Universe **8** (2022), no. 8, 404, doi:10.3390/universe8080404.

[7] ———, *The quantization of gravity: The quantization of the full Einstein equations*, Symmetry **15** (2023), no. 8, 1599, doi:10.3390/sym15081599.

[8] ———, *The Quantization of Gravity*, 2nd ed., Fundamental Theories of Physics, vol. 194, Springer, Cham, November 2024, doi:10.1007/978-3-031-67922-3.

[9] Erich Kamke, *Gewöhnliche Differentialgleichungen*, 10.Aufl., unveränd. Nachdr. d. 8., durchges. Aufl. ed., Teubner, Stuttgart, 1983 (ger).

[10] Krzysztof Maurin, *Methods of Hilbert spaces*, Translated from the Polish by Andrzej Alexiewicz and Waclaw Zawadowski. Monografie Matematyczne, Tom 45, Państwowe Wydawnictwo Naukowe, Warsaw, 1967.

[11] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge; New York; Melbourne, 2010 (eng).

Ruprecht-Karls-Universität, Institute for Mathematics, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

Email address: gerhardt@math.uni-heidelberg.de

URL: https://www.math.uni-heidelberg.de/studinfo/gerhardt/