# Fundamental Theories of Physics

Volume 194

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Claus Gerhardt

# The Quantization of Gravity

Second Edition

Springer

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ISSN 0168-1222 ISSN 2365-6425 (electronic)
Fundamental Theories of Physics
ISBN 978-3-031-67921-6 ISBN 978-3-031-67922-3 (eBook)
https://doi.org/10.1007/978-3-031-67922-3

Mathematics Subject Classification: 83, 83C, 83C45

1st edition: © Springer International Publishing AG, part of Springer Nature 2018, corrected publication 2018
2nd edition: © The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2024

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# Preface to the Second Edition

As we pointed out in the Preface of the First Edition we want to quantize the full Einstein equations and solve the resulting equation by using separation of variables to express the solutions $u$ by a product of eigenfunctions or eigendistributions of self-adjoint operators in corresponding Hilbert spaces. For the quantization, we work in a fiber bundle $E$ the base space of which is a Cauchy hypersurface $(\mathcal{S}_0, \bar{\sigma}_{ij})$ of the quantized spacetime, where $\bar{\sigma}_{ij}$ is the induced metric. The fibers $F(x)$ over $x \in \mathcal{S}_0$ are Riemannian metrics $g_{ij}(x)$ if external fields are excluded. In an appropriate local trivialization, we obtained a coordinate system $(\xi^a)$, $0 \le a \le m$,

$$m = \frac{(n-1)(n+2)}{2}$$

$n = \dim \mathcal{S}_0$, such that the metrics $g_{ij}$ can be written

$$g_{ij} = t^{\frac{4}{n}} \sigma_{ij},$$

where

$$0 < t = \xi^0 < \infty$$

and the metric $\sigma_{ij}$ belongs to the hypersurface or subbundle

$$M = \{t = 1\} \subset E.$$

The solutions $u$ then depend on the variables $(t, \sigma_{ij}, x)$, where $\sigma_{ij}$ does not depend on $t$ and $t$ not on $x$. We refer to $t$ as quantum time and $x$, $\sigma_{ij}$ as spatial variables.

In the papers [32, 35], written since the publication of the first edition, we solved the eigenfunction problems and we could express $u$ in the form

$$u = w\hat{v}v,$$

where $w = w(t)$ is the temporal eigenfunction, $\hat{v} = \hat{v}(\sigma_{ij}(x))$ can be identified with an eigenfunction of the Laplacian of the symmetric space

$$X = SL(n, \mathbb{R})/SO(n)$$

such that

$$\hat{v}(\bar{\sigma}_{ij}(x)) = 1 \quad \forall x \in \mathcal{S}_0,$$

where $\bar{\sigma}_{ij}$ is the fixed induced metric of $\mathcal{S}_0$. The eigenfunctions $\hat{v}$ represent the elementary gravitons corresponding to the degrees of freedom in choosing the entries of Riemannian metrics with determinants equal to one. These are all the degrees of freedom available because of the coordinate system invariance: For any smooth Riemannian metric there exists an atlas such that the determinant of the metric is equal to one, Lemma 3.2.1.

Finally, the function $v$ is an eigenfunction of an essentially self-adjoint differential operator in $\mathcal{S}_0$.

The temporal eigenfunction was at first only a solution of an ODE but in this second edition, we could also prove, for arbitrary $n \geq 3$, that $w$ is an eigenfunction of a self-adjoint operator in $\mathbb{R}_+$ with a pure point spectrum provided the cosmological constant $\Lambda$ is negative and in case $n = 3$ also for $\Lambda > 0$.

In Chap. 5 we treat the quantization of gravity combined with the forces of the Standard Model if $n = 3$, but we could only solve the Wheeler-DeWitt equation, which is the result of the quantization of the Hamilton condition representing only the normal Einstein equation and does not include the tangential Einstein equations as well. This chapter is based on our paper [33].

Heidelberg, Germany
May 2024

Claus Gerhardt

# Preface to the First Edition

A unified quantum theory incorporating the four fundamental forces of nature is one of the major open problems in physics. The Standard Model combines electromagnetism, the strong force and the weak force, but ignores gravity. The quantization of gravity is therefore a necessary first step to achieve a unified quantum theory.

The Einstein equations are the Euler-Lagrange equations of the Einstein-Hilbert functional and quantization of a Lagrangian theory requires to switch from a Lagrangian view to a Hamiltonian view. In a ground breaking paper, Arnowitt, Deser and Misner [2] expressed the Einstein-Hilbert Lagrangian in a form which allowed to derive a corresponding Hamilton function by applying the Legendre transformation. However, since the Einstein-Hilbert Lagrangian is singular, the Hamiltonian description of gravity is only correct if two additional constraints are satisfied, namely, the Hamilton constraint, which is expressed by the equation $H = 0$, where $H$ is the Hamilton function, and the diffeomorphism constraint. Dirac [12] proved how to quantize a constrained Hamiltonian system—at least in principle—and his method has been applied to the Hamiltonian setting of gravity, cf. the paper of DeWitt [10] and the monographs by Kiefer [54] and Thiemann [63]. In the general case, when arbitrary globally hyperbolic spacetime metrics are allowed, the problem turned out to be extremely difficult and solutions could only be found by assuming a high degree of symmetry, cf., e.g., [20].

However, in [23, 24, 21] we proposed a model for the quantization of gravity for general hyperbolic spacetimes, in which we eliminated the diffeomorphism constraint by reducing the number of variables and proving that the Euler-Lagrange equations for this special class of metrics were still the full Einstein equations. The Hamiltonian description of the Einstein-Hilbert functional then allowed a canonical quantization. We quantized the action by looking at the Wheeler-DeWitt equation in a fiber bundle $E$, where the base space is a Cauchy hypersurface of the spacetime which has been quantized and the elements of the fibers are Riemannian metrics. The fibers of $E$ are equipped with a Lorentzian metric such that they are globally hyperbolic and the transformed Hamiltonian, which is now a hyperbolic operator,

$\hat{H}$, is a normally hyperbolic operator acting only in the fibers. The Wheeler-DeWitt equation has the form $\hat{H}u = 0$ with $u \in C^\infty(E, \mathbb{C})$ and we defined with the help of the Green's operator a symplectic vector space and a corresponding Weyl system.

The Wheeler-DeWitt equation seems to be the obvious quantization of the Hamilton condition. However, $\hat{H}$ acts only in the fibers and not in the base space which is due to the fact that the derivatives are only ordinary covariant derivatives and not functional derivatives, though they are supposed to be functional derivatives, but this property is not really invoked when a functional derivative is applied to $u$, since the result is the same as applying a partial derivative.

Therefore, we discarded the Wheeler-DeWitt equation in [25] and expressed the Hamilton condition differently by looking at the evolution equation of the mean curvature of the foliation hypersurfaces $M(t)$ and implementing the Hamilton condition on the right-hand side of this evolution equation. The left-hand side, a time derivative, we replaced by the corresponding Poisson brackets. After canonical quantization, the Poisson brackets became a commutator and now we could employ the fact that the derivatives are functional derivatives, since we had to differentiate the scalar curvature of a metric. As a result, we obtained an elliptic differential operator in the base space, the main part of which was the Laplacian of the metric.

On the right-hand side of the evolution equation the interesting term was $H^2$, the square of the mean curvature. It transformed to a second time derivative, the only remaining derivative with respect to a fiber variable, since the differentiations with respect to the other variables canceled each other. The resulting quantized equation is then a wave equation in a globally hyperbolic spacetime

$$Q = (0, \infty) \times \mathcal{S}_0,$$

where $\mathcal{S}_0$ is the Cauchy hypersurface. When $\mathcal{S}_0$ is a space of constant curvature than the wave equation, considered only for functions $u$ which do not depend on $x$, is identical to the equation obtained by quantizing the Hamilton constraint in a Friedmann universe without matter but including a cosmological constant.

There also exist temporal and spatial self-adjoint operators $H_0$ resp. $H_1$ such that the hyperbolic equation is equivalent to

$$H_0 u - H_1 u = 0,$$

where $u = u(t, x)$, and $H_0$ has a pure point spectrum with eigenvalues $\lambda_i$ while, for $H_1$, it is possible to find corresponding eigendistributions for each of the eigenvalues $\lambda_i$, if $\mathcal{S}_0$ is asymptotically Euclidean or if the quantized spacetime is a black hole with a negative cosmological constant, cf. [28, 27, 30]. The hyperbolic equation then has a sequence of smooth solutions which are products of temporal eigenfunctions and spatial eigendistributions. Due to this "spectral resolution" of the wave equation, we were also able to apply quantum statistics to the quantized systems, cf. [29]. These quantum statistical results could help to explain the nature of dark matter and dark energy.

We believe that the wave equation model in the spacetime $Q$ is a very promising model for describing quantum gravity.

Heidelberg, Germany | Claus Gerhardt
December 2017

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