Preface to the second edition

As we pointed out in the preface of the first edition we want to quantize the full Einstein equations and solve the resulting equation by using separation of variables to express the solutions u by a product of eigenfunctions or eigendistributions of self-adjoint operators in corresponding Hilbert spaces. For the quantization we work in a fiber bundle E the base space of which is a Cauchy hypersurface $(\mathcal{S}_0, \bar{\sigma}_{ij})$ of the quantized spacetime, where $\bar{\sigma}_{ij}$ is the induced metric. The fibers $F(x)$ over $x \in \mathcal{S}_0$ are Riemannian metrics $g_{ij}(x)$ if external fields are excluded. In an appropriate local trivialization we obtained a coordinate system (ξ^a), $0 \le a \le n$, $n = \dim \mathcal{S}_0$, such that the metrics g_{ij} can be written

$$g_{ij} = t^{\frac{4}{n}} \sigma_{ij},$$

where

$$0 < t = \xi^0 < \infty$$

and the metric σ_{ij} belongs to the hypersurface or subbundle

$$M = \{t = 1\} \subset E.$$

The solutions u then depend on the variables (t, σ_{ij}, x), where σ_{ij} does not depend on t and t not on x. We refer to t as quantum time and x, σ_{ij} as spatial variables.

In the papers [**32, 35**], written since the publication of the first edition, we solved the eigenfunction problems and we could express u in the form

$$u = w\hat{v}v,$$

where $w = w(t)$ is the temporal eigenfunction, $\hat{v} = \hat{v}(\sigma_{ij}(x))$ can be identified with an eigenfunction of the Laplacian of the symmetric space

$$X = SL(n, \mathbb{R})/SO(n)$$

such that

$$\hat{v}(\bar{\sigma}_{ij}(x)) = 1 \qquad \forall\, x \in \mathcal{S}_0,$$

where $\bar{\sigma}_{ij}$ is the fixed induced metric of $\mathcal{S}_0$. The eigenfunctions $\hat{v}$ represent the elementary gravitons corresponding to the degrees of freedom in choosing the entries of Riemannian metrics with determinants equal to one. These are all the degrees of freedom available because of the coordinate system invariance: For any smooth Riemannian metric there exists an atlas such that the determinant of the metric is equal to one, cf. Lemma 3.2.1 on page 64.

Finally, the function v is an eigenfunction of an essentially self-adjoint differential operator in $\mathcal{S}_0$.

The temporal eigenfunction was at first only a solution of an ODE but in this second edition we could also prove, for arbitrary $n \geq 3$, that w is an eigenfunction of a self-adjoint operator in $\mathbb{R}_+$ with a pure point spectrum provided the cosmological constant Λ is negative and in case $n = 3$ also for $\Lambda > 0$.

In Chapter 5 we treat the quantization of gravity combined with the forces of the Standard Model if $n = 3$, but we could only solve the Wheeler-DeWitt equation, which is the result of the quantization of the Hamilton condition representing only the normal Einstein equation and does not include the tangential Einstein equations as well. This chapter is based on our paper [**33**].

Heidelberg, May 2024 Claus Gerhardt