EFFECTIVE HAUSDORFF DIMENSION

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ABSTRACT. We continue the study of effective Hausdorff dimension as it was initiated by LUTZ. Whereas he uses a generalization of martingales on the Cantor space to introduce this notion we give a characterization in terms of effective s-dimensional Hausdorff measures, similar to the effectivization of Lebesgue measure by MARTIN-LÖF. It turns out that effective Hausdorff dimension allows to classify sequences according to their 'degree' of algorithmic randomness, i.e., their algorithmic density of information. Earlier the works of STAIGER and RYABKO showed a deep connection between Kolmogorov complexity and Hausdorff dimension. We further develop this relationship and use it to give effective versions of some important properties of (classical) Hausdorff dimension. Finally, we determine the effective dimension of some objects arising in the context of computability theory, such as degrees and spans.

1. INTRODUCTION

Generally speaking, the concepts of Hausdorff measure and dimension are a generalization of Lebesgue measure theory. In the early 20th century, HAUSDORFF [9] used the classical Caratheodory construction of measures to define a whole family of outer measures. For examining a set of a peculiar topological or geometrical nature Lebesgue measure often is too coarse to investigate the features of the set, so one may 'pick' a measure from this family of outer measures that is suited to study this particular set. This is one reason why Hausdorff measure and dimension became a prominent tool in fractal geometry.

Hausdorff dimension is extensively studied in the context of dynamical systems, too. On the Cantor space, the space of all infinite binary sequences, which itself can be regarded as a "fractal" (it is homeomorphic to the well known middle-third Cantor set in the unit interval), the interplay between dimension and concepts from dynamical systems such as entropy becomes really close. Results of BESICOVITCH [3] and EGGLESTON [7] early brought forth a correspondence between the Hausdorff dimension of frequency sets (i.e., sets of sequences in which every symbol occurs with a certain frequency) and the entropy of a process creating such sequences as typical outcomes. Besides, under certain conditions the Hausdorff dimension of a set in the Cantor space equals the topological entropy of this set, viewed as a shift space.

An effective version of measure and entropy has been developed since the middle of the 20th century. MARTIN-LÖF [14] effectivized the notion of a Lebesgue nullset in order to characterize objects (sequences) that are algorithmically random (namely those that do not have

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effective measure 0). The theory of *Kolmogorov complexity* (see [11] for a thorough introduction), on the other hand, can be regarded as an effective version of entropy, which makes it possible to determine the entropy of individual objects, just as Martin-Löf randomness can declare individual sequences as random. And indeed, ways emerged how to characterize randomness in terms of Kolmogorov complexity.

Nevertheless, the border between randomness and non-randomness is quite stiff. The theory of Martin-Löf randomness offers no possibility to distinguish between different "degrees" of randomness. It might be that one sequence behaves "close to random" (for instance by satisfying a lot of statistical laws) although it is not, whereas other nonrandom sequences are very regular. In contrast to this, the notion of entropy can be interpreted to describe the degree of randomness of a dynamical system. Kolmogorov complexity, as an algorithmic version of entropy, does the same for finite binary sequences.

By developing an effective version of Hausdorff measure and dimension, one may hope to obtain a tool for classifying sets and sequences according to their degree of algorithmic randomness. Besides, as Hausdorff dimension is also a geometrical notion (it is invariant under bi-Lipschitz transformations), effectivizing the theory might point new techniques coming from fractal geometry for use in algorithmic measure and information theory.

These ideas, of course, are not entirely new. BRUDNO [4] and WHITE [22] studied the relationship between the entropy of a symbolic dynamical system and the Kolmogorov complexity of an individual trajectory of a system. RYABKO [19], STAIGER [20] and CAI and HARTMANIS [5] observed close links between the Hausdorff dimension of a set and the Kolmogorov complexity of its members. LUTZ [13] was the first to explicitly define an effective notion of Hausdorff dimension. He also introduced a resource bounded version ([12], see also [1]).

In this article, we further develop effective Hausdorff measure and dimension along the line of LUTZ [12, 13]. However, we do not follow his martingale approach and develop a Martin-Löf style definition instead. The outline of the paper is as follows. In Section 2 we give a short overview over classical Hausdorff dimension on the Cantor space. In Section 3 we present effective Hausdorff measure and dimension, along with effective versions of some of their important properties. To achieve this, the close connection of effective dimension to Kolmogorov complexity will be used. In Section 4 we determine the effective dimension of some objects arising in the context of computability theory, such as degrees or spans.

NOTATION. Our notation is fairly standard. $\{0,1\}^\infty$ denotes the *Cantor space*, the set of all infinite binary sequences. The greek letters ζ, η, ξ and ω denote elements of the Cantor space. We write $\xi(n)$, $n \in \mathbb{N}$, to denote the n-th bit of the sequence ξ, and $\xi|_n$ denotes the n-bit initial segment of ξ, that is $\xi|_n = \xi(0)\xi(1)\ldots\xi(n-1)$. We identify subsets of the natural numbers with their characteristic sequences, so sometimes we will regard them as elements of the Cantor space, too. Therefore, subsets of the Cantor space are also called *classes*. The lower case roman letters i, j, k, m, n denote natural numbers, whereas v, w, x, y, z usually denote finite binary strings; $l(x)$ denotes the length of a string, so $x = x(0)\dots x(l(x)-1)$, and $\{0,1\}^*$ denotes the set of all finite binary strings. We write $x \prec y$ if x is an initial segment of y, i.e., $l(x) < l(y)$ and $\forall i < l(x) : x(i) = y(i)$. $x \prec \xi$, $\xi \in \{0,1\}^\infty$, is defined analogously. We also say that ξ and y *extend* x.

Furthermore, we assume some familiarity with the basic concepts of computability theory such as recursive and recursively enumerable sets, reducibilities, degrees and spans. For an extensive treatment, see the textbook by ODIFREDDI [17].

2. CLASSICAL HAUSDORFF DIMENSION

The basic idea behind Hausdorff dimension is to generalize the process of measuring a set by approximating (covering) it with sets whose measure is already known. Especially, the size of the sets used in the measurement process will be manipulated by certain transformations, thus making it harder or easier to approximate a set with a covering of small accumulated measure. This gives rise to the notion of *Hausdorff measure*.

We will introduce this notion on the Cantor space $\{0,1\}^\infty$ directly, where we can make use of some of its special features in order to simplify some definitions. For a general treatment of Hausdorff dimension and measure on metric or measure spaces, see the textbooks by EDGAR [6], FALCONER [8] or MATTILA [15].

We endow the Cantor space with the usual metric d for sequences. For two sequences $\xi, \omega \in \{0,1\}^\infty$, define $c(\xi,\omega)$ to be their *maximal common initial segment*. Now let

$$d(\xi,\omega) = 2^{-l(c(\xi,\omega))}.$$

We write $l(\xi,\omega)$ for $l(c(\xi,\omega))$. The *diameter* $d(X)$ of a class $X \subseteq \{0,1\}^\infty$ is defined by

$$d(X) = \sup\{d(\xi,\omega) : \xi,\omega \in X\}$$

The standard topology of $\{0,1\}^\infty$ is generated by the *basic open cylinders*:

$$C_w = \{\xi \in \{0,1\}^\infty : w \prec \xi\}, \ \ w \in \{0,1\}^*$$

Assigning each of these cylinders the measure

$$\lambda(C_w) = 2^{-l(w)} = d(C_w)$$

induces the Lebesgue measure λ on $\{0,1\}^\infty$, which is measure-theoretically isomorphic to the standard Lebesgue measure on $\{r : 0 \le r < 1\}$, the unit interval.

Now we can introduce Hausdorff measures. Let $X \subseteq \{0,1\}^\infty$, $\delta > 0$. A (countable) family $\{C_{w_i}\}_{i\in\mathbb{N}}$ is a *δ-cover* of X, if $(\forall i)\ d(C_{w_i}) \le \delta$ and $X \subseteq \bigcup C_{w_i}$. For $s \ge 0$, define

$$\mathcal{H}^s_\delta(X) = \inf\{\sum_{i\in\mathbb{N}} d(C_{w_i})^s : \{C_{w_i}\} \text{ is a } \delta\text{-cover of } .X\}$$

As δ decreases, there are fewer δ-covers available, hence $\mathcal{H}^s_\delta$ is non-decreasing. Consequently, the value

$$\mathcal{H}^s(X) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(X)$$

is well defined, but may be infinite.

$\mathcal{H}^s(X)$ is called the *s-dimensional Hausdorff measure* of X. It can be shown that $\mathcal{H}^s$ is an outer measure and that the Borel sets of $\{0,1\}^\infty$ are $\mathcal{H}^s$-measurable. Of course, for $s = 1$ we get the usual Lebesgue outer measure.

The outer measures $\mathcal{H}^s$ have an important property.

Proposition 2.1. *Let $X \subseteq \{0,1\}^\infty$. If, for some $s \geq 0$, $\mathcal{H}^s(X) < \infty$, then $\mathcal{H}^t(X) = 0$ for all $t > s$.*

Proof. Let $\mathcal{H}^s(X) < \infty$, $t > s$. If $\{C_{w_i}\}$ is a δ-cover of X, $\delta > 0$, we have

$$\sum_{i\in\mathbb{N}} d(C_{w_i})^t \leq \delta^{t-s} \sum_{i\in\mathbb{N}} d(C_{w_i})^s$$

so, taking infima, $\mathcal{H}^t_\delta(X) \leq \delta^{t-s}\mathcal{H}^s_\delta(X)$. As $\delta \to 0$, the result follows. □

This means that there can exist only one point $s \geq 0$ where a given class might have finite positive s-dimensional Hausdorff measure. This point is the *Hausdorff dimension* of the class.

Definition 2.2. For a class $X \subseteq \{0,1\}^\infty$, define the *Hausdorff dimension* of X as

$$\dim_{\mathrm{H}}(X) = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

In the following, we list some characteristic properties of Hausdorff dimension.

Refinement of measure 0. If $\lambda(X) \neq 0$ then $\dim_{\mathrm{H}}(X) = 1$. This follows from the fact that $\mathcal{H}^1$ is the Lebesgue outer measure. In particular, $\mathcal{H}^1(\{0,1\}^\infty) = \lambda(\{0,1\}^\infty) = 1$. On the other hand, no $X \subseteq \{0,1\}^\infty$ can have Hausdorff dimension greater than 1, as $\mathcal{H}^s(X) = 0$ for all $s > 1$. (This can be seen by taking the 'trivial' covering consisting for every $\delta = 1/2^n$ of all cylinders C_w with $l(w) = n$.)

Monotonicity. If $X \subseteq Y$ then $\dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{H}}(Y)$.

Stability. For $X, Y \subseteq \{0,1\}^\infty$ we have

$$\dim_{\mathrm{H}}(X \cup Y) = \max\{\dim_{\mathrm{H}}(X), \dim_{\mathrm{H}}(Y)\}.$$

This can be generalized to the case of countable unions.

Countable Stability. Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable family of classes. Then

$$\dim_{\mathrm{H}}\Big(\bigcup_{i\in\mathbb{N}} X_i\Big) = \sup_{i\in\mathbb{N}}\{\dim_{\mathrm{H}}(X_i)\}.$$

Geometric Invariance. Let $X \subseteq \{0,1\}^\infty$ and $f : X \to \{0,1\}^\infty$ be a bi-Lipschitz transformation, i.e., there exists $c_1, c_2 > 0$ such that $c_1 d(\xi,\omega) \leq d(f(\xi), f(\omega)) \leq c_2 d(\xi,\omega)$ for all $\xi, \omega \in X$. Then $\dim_{\mathrm{H}}(f(X)) = \dim_{\mathrm{H}}(X)$. This property follows easily from the behaviour of Hausdorff measure / dimension under *Hölder transformations*.

Proposition 2.3. *Let $X \subseteq \{0,1\}^\infty$. Suppose $f : X \to \{0,1\}^\infty$ satisfies a Hölder condition: There exist $c, \alpha > 0$ such that $d(f(\xi), f(\omega)) \leq c d(\xi,\omega)^\alpha$ for all $\xi, \omega \in X$. Then*

$$\dim_{\mathrm{H}}(f(X)) \leq \tfrac{1}{\alpha} \dim_{\mathrm{H}}(X).$$

The primary interest, of course, now lies in determining the Hausdorff dimension of classes and in exposing the structure of sets having non-integral dimension. However, the main obstacle for a direct applicability of Hausdorff dimension in the area of computability theory is the property of countable stability, which easily implies that all countable classes have dimension 0. Therefore, as in the case of effective measure theory, the notion of Hausdorff dimension first has to be effectivized, which is done in the next section.

3. EFFECTIVE HAUSDORFF DIMENSION

The effectivization of Hausdorff dimension resembles the effectivization of Lebesgue measure on $\{0,1\}^\infty$, as it was done by MARTIN-LÖF [14] in order to characterize algorithmic randomness. As measure 0 is defined via coverings the crucial step lies in allowing effective, that is recursively enumerable coverings only. Hausdorff measure is an outer measure, defined via coverings as well. Therefore, the same strategy may be applied in effectivizing Hausdorff dimension, using the following alternative characterization of s-dimensional measure 0 (a special version of Theorem 32 in [18]).

Proposition 3.1. *A class $X \subseteq \{0,1\}^\infty$ has s-dimensional Hausdorff measure* 0 *if and only if there exists a set $C \subseteq \{0,1\}^*$ such that*

$$\sum_{w\in C} 2^{-l(w)s} < \infty \quad \textit{and} \quad (\forall \xi \in X)\,(\exists^\infty w \in C)\,[w \prec \xi]. \tag{1}$$

Proof. Suppose $\mathcal{H}^s(X) = 0$. Then, for all n, there is a set $C_n \subseteq \{0,1\}^*$ that is a 2^{-n}-covering of X and for which $\sum_{w\in C_n} 2^{-l(w)s} < 2^{-n}$ holds (since $\mathcal{H}^s_{2^{-n}}(X) = 0$). Set $C = \bigcup C_n$. Then we have $\sum_{w\in C} 2^{-l(w)s} < 1$ (absolute convergence) and each $\xi \in X$ extends infinitely many $w \in C$.

Now suppose that there is a set $C = \{w_0, w_1, w_2, \dots\}$ with $\sum_{w\in C} 2^{-l(w)s} < \infty$ and for all $\xi \in X$ there are infinitely many n such that $w_n \prec \xi$. Let m be a natural number. It suffices to show that for any $\varepsilon > 0$ there exists a 2^{-m}-cover W of X such that $\sum_{w\in W} 2^{-l(w)s} < \varepsilon$.

Choose N so large that

$$\sum_{n\geq N} 2^{-l(w)s} < \varepsilon \text{ and for all } n \geq N,\, l(w_n) \geq m.$$

As each $\xi \in X$ extends infinitely many w_n, for each $\xi \in X$ there exists some $n \geq N$ (in fact, infinitely many n) such that $w_n \prec \xi$. Thus $\{w_n : n \geq N\}$ is a 2^{-m}-cover of X with the desired properties. □

We call a set C that satisfies (1) an *s-covering set*.

It is quite obvious now how to define effective Hausdorff measure 0.

Definition 3.2. Let $s \geq 0$. A class $X \subseteq \{0,1\}^\infty$ has *effective s-dimensional Hausdorff measure* 0, $\mathcal{H}^{1,s}(X) = 0$, if there exists a recursively enumerable set C that satisfies (1).

In this definition it is not presupposed that s is computable. However, as we go on to investigate effective versions of Hausdorff measure and dimension, often it will be necessary that s is in somer form given effectively, too. Therefore (for sake of simplicity), from now on we concentrate on rational s, which is sufficient, as $\mathbb{Q}$ lies dense in $\mathbb{R}$.

Furthermore, note that this approach to effective measure is not the usual one, as it was first proposed by MARTIN-LÖF. Definition 3.2 is a generalization of SOLOVAY's approach to randomness. Nevertheless, the following lemma ensures that, as in the case of effective Lebesgue measure, the two approaches lead to the same concept.

Lemma 3.3. *Let $X \subseteq \{0,1\}^\infty$, $s \in \mathbb{Q}$. Then $\mathcal{H}^{1,s}(X) = 0$ if and only if there is a recursive sequence $C_1, C_2, C_3, \ldots$ of r.e. sets of strings such that for all n*

$$(\forall \xi \in X)\,(\exists w \in C_n)\, w \prec \xi \;\text{ and }\; \sum_{w \in C_n} 2^{-l(w)s} \leq 2^{-n}. \tag{2}$$

So, in particular, $\mathcal{H}^{1,1}(X) = 0$ means that X is an effective nullclass in the sense of MARTIN-LÖF. We denote this special case by $\lambda^1(X) = 0$.

It is easy to see that the following 'effective' version of Proposition 2.1 holds.

Proposition 3.4. *Let $X \subseteq \{0,1\}^\infty$. If, for some $s \geq 0$, $\mathcal{H}^{1,s}(X) = 0$, then $\mathcal{H}^t(X) = 0$ for all $t > s$, too.*

The definition of effective Hausdorff dimension follows in a straightforward way.

Definition 3.5. The *effective Hausdorff dimension* of a class $X \subseteq \{0,1\}^\infty$ is defined as

$$\dim^1_{\mathrm{H}}(X) = \inf\{s \geq 0 : \mathcal{H}^{1,s}(X) = 0\}.$$

We check some basic properties of effective dimension.

Dimension Conservation. We have $\dim^1_{\mathrm{H}}(\{0,1\}^\infty) = 1$. Obviously, the trivial cover $C = \{0,1\}^*$ is r.e. and $\sum_{w \in \{0,1\}^*} 2^{-l(w)s} < \infty$ if $s > 1$.

Monotonicity. $\dim^1_{\mathrm{H}}(X) \leq \dim^1_{\mathrm{H}}(Y)$ for $X \subseteq Y$ follows just as in the non-effective case.

Refinement of effective Lebesgue measure 0. $\lambda^1(X) \neq 0 \Rightarrow \dim^1_{\mathrm{H}}(X) = 1$. This is another straightforward analogy to the classical case.

Classical and effective Hausdorff dimension. $\dim^1_{\mathrm{H}}(X) \geq \dim_{\mathrm{H}}(X)$ follows directly from the definition.

The other important properties of Hausdorff dimension, countable stability and invariance under bi-Lipschitz transformations, require more careful treatment.

One of the great advantages of effective measure is the existence of a maximal effective nullclass, i.e., one that contains all other effective nullclasses. This has as an easy corollary the closure of effective nullclasses under countable unions. In order to prove a similar property for $\mathcal{H}^{1,s}$-measure, one has to take into account that s is a real number, i.e. , might not be computable. However, for rational s, we do not face any problems. So, using Lemma 3.3 and adapting the usual construction of a maximal nullclass, we can prove the following.

Proposition 3.6. *Let $s \geq 0$ be rational. There exists a maximal $\mathcal{H}^{1,s}$-nullclass, i.e., some U_s such that for all $X \subseteq \{0,1\}^\infty$,*

$$\mathcal{H}^{1,s}(X) = 0 \ \Leftrightarrow \ X \subseteq U_s.$$

Obviously, Proposition 3.6 yields the countable stability of effective dimension.

Besides, it now makes sense to consider the effective dimension of an individual sequences (viewed as a singleton class), as these have not automatically effective dimension 0. (In the following, we write $\dim^1_{\mathrm{H}}(\xi)$ for $\dim^1_{\mathrm{H}}(\{\xi\})$, $\xi \in \{0,1\}^\infty$.)

An example are the *Martin-Löf random sequences*. These are precisely the sequences not contained in the maximal λ^1-nullclass. Every single Martin-Löf random sequence has effective dimension 1.

Furthermore, the effective dimension of a class can be characterized in terms of the effective dimension of its members. The following theorem has first been proved by LUTZ [13].

Theorem 3.7 (LUTZ). *For any class $X \subseteq \{0,1\}^\infty$,*

$$\dim^1_{\mathrm{H}}(X) = \sup_{\xi \in X} \dim^1_{\mathrm{H}}(\xi).$$

As regards an effective version of bi-Lipschitz invariance, the problem, of course, lies in the fact that Hölder transformations of the Cantor space need not to be effective, which means that an effective covering of some class does not automatically yield an effective covering of its image. Easy objects could be transformed in to very complicated ones from a computability point of view.

In order to get results of a similar flavour as those of Proposition 2.3 we have to take into account the computational behaviour as well as the geometrical behaviour of a mapping. In this setting, Kolmogorov complexity will prove quite useful.

There exist different versions of Kolmogorov complexity, 'plain' complexity C and prefix complexity K. (Sometimes, these are also referred to as K and H, respectively.) Since we are interested in their asymptotical behaviour, which is equal (i.e. , $C(x) \leq K(x) \leq C(x) + 2\log C(x)$), it does not matter which concept we use. As effective Lebesgue measure and randomness can be approached via prefix complexity K, we stick to this notion.

Let us fix a universal prefix-free Turing machine U, that is, a machine for which no two halting inputs are prefixes of one another and that can simulate all other prefix-free Turing machines. We define

$$K(x|y) = K_U(x|y) = \min\{l(p) : \ U(p,y) = x\}$$

and

$$K(x) = K(x|\epsilon)$$

where ϵ denotes the empty string. We do not go into details of this theory here, the interested reader may consult the comprehensive book by LI and VITANYI [11].

In the context of Hausdorff dimension, an important feature of prefix complexity is its characterization as a universal recursively enumerable semimeasure (see [23]). This characterization might be interpreted in the following way: The (prefix) Kolmogorov complexity of a finite initial segment of a sequence tells us how well this sequence at this level can be described by an recursively enumerable process, or, to put it another way, individuated from the other possible strings of the same length. This leads to a nice characterization of effective dimension. For $\xi \in \{0,1\}^\infty$, define $\underline{K}(\xi) = \liminf_{n\to\infty} K(\xi|_n)/n$.

Theorem 3.8. *For any* $\xi \in \{0,1\}^\infty$,

$$\dim^1_{\mathrm{H}}(\xi) = \underline{K}(\xi). \tag{3}$$

The proof of Theorem 3.8 was first given by LUTZ [13] and MAYORDOMO [16], but much of it is inherent in the works of RYABKO [19], STAIGER [20] and CAI and HARTMANIS [5].

LUTZ showed that $\underline{K}(\xi) \leq \dim^1_{\mathrm{H}}(\xi) \leq \overline{K}(\xi) \stackrel{\text{def}}{=} \limsup K(\xi|_n)/n$, using a martingale characterization of effective dimension (which he introduced as *constructive dimension*). The first author has given an alternative proof using an effectivized *mass distribution principle* (see [8] for this technique).

We give a direct proof of the inequality $\dim^1_{\mathrm{H}}(\xi) \leq \underline{K}(\xi)$.

Proof (of Theorem 3.8). Let $s > \underline{K}(\xi)$ be rational. We show that for such s, $\mathcal{H}^{1,s}(\xi) = 0$. Note that

$$C = \{w \in \{0,1\}^* : K(w) < sl(w)\}$$

is a recursively enumerable set. We claim that it is also an s-covering sequence for ξ, i.e., satisfies (1) for $\{\xi\}$. First, there are infinitely many n such that $\xi|_n \in C$, for $s > \underline{K}(\xi)$. Second, it is easy to see that

$$\sum_{w\in C} 2^{-l(w)s} < \infty,$$

for $w \in C$ implies $K(w) < sl(w)$ and therefore $2^{-K(w)} > 2^{-l(w)s}$, and we know that $\sum 2^{-K(w)}$ converges, due to the Kraft-Chaitin inequality (see [11]). □

Remark 3.9. It is possible to characterize Hausdorff dimension in terms of martingales. This is the approach chosen by LUTZ[13]. A *martingale* m is a function mapping binary strings to positive rationals that satisfies $m(w0) + m(w1) = 2m(w)$ for all $w \in \{0,1\}^*$. One can compute a weighted sum over all martingales for which the set $\{(w,c) : w \in \{0,1\}^*, c \in \mathbb{Q}, m(w) < c\}$ is recursively enumerable and obtain a maximal martingale m_0 (see , for instance, [13] for details). Now the effective Hausdorff dimension of a class C is

$$\dim^1_{\mathrm{H}}(C) = \inf\{s : (\forall\xi \in C)(\exists^\infty n) m_0(\xi|_n) > 2^{n(1-s)}\}.$$

Now we can start investigating the behaviour of effective dimension under transformations. Let $f : \{0,1\}^\infty \to \{0,1\}^\infty$ be a Hölder transformation on the Cantor space, i.e.,

$$\forall \xi,\omega \in \{0,1\}^\infty \; d(f(\xi), f(\omega)) \leq c\, d(\xi,\omega)^\alpha \tag{4}$$

for some $\alpha, c > 0$. This implies (recall the definition of metric d)

$$l(f(\xi), f(\omega)) \geq \alpha\, l(\xi, \omega) - \log c$$

Indeed, one can exploit the special structure of the Cantor space to study a more general class of mappings based on functions operating on strings. Call a mapping $\varphi : \{0,1\}^* \to \{0,1\}^*$ *α-expansive*, $\alpha > 0$, if

(I) $x \preceq y \Rightarrow \varphi(x) \preceq \varphi(y)$ (φ is *monotone*),
(II) $\forall \omega \in \{0,1\}^\infty$ $\liminf_{n\to\infty} l(\varphi(\omega|_n))/n \geq \alpha$.

Obviously, property (II) implies that $\forall \omega \in \{0,1\}^\infty$ $\lim_{n\to\infty} l(\varphi(\omega|_n)) = \infty$, so (I) and (II) induce a mapping $\hat{\varphi} : \{0,1\}^\infty \to \{0,1\}^\infty$. One can show that for each Hölder transformation $f : \{0,1\}^\infty \to \{0,1\}^\infty$ there exists an α-expansive φ such that $\hat{\varphi} = f$. On the other hand, not every function $\hat{(\varphi)}$ induced by a mapping φ satisfying (I) and (II) is necessarily Hölder.

Theorem 3.8 enables us to analyze the effective dimension of $\hat{\varphi}(\xi)$ by investigating the asymptotical behaviour of the Kolmogorov complexity $K(\varphi(\xi|_n))$ of its initial segments. Note that, if $\varphi : \{0,1\}^* \to \{0,1\}^*$ is recursive, then $K(\varphi(x)) \leq K(x)$. (CAI and HARTMANIS [5] gave a similar result for classical Hausdorff dimension.)

Theorem 3.10. *Let $\varphi : \{0,1\}^* \to \{0,1\}^*$ be a recursive α-expansive mapping. Then it holds that, for any $X \subseteq \{0,1\}^\infty$,*

$$\dim^1_{\mathrm{H}}(\hat{\varphi}(X)) \leq \tfrac{1}{\alpha} \dim^1_{\mathrm{H}}(X). \tag{5}$$

Proof. It suffices to show that, for any $\xi \in X$,

$$\dim^1_{\mathrm{H}}(\hat{\varphi}(\xi)) \leq \tfrac{1}{\alpha} \dim^1_{\mathrm{H}}(\xi).$$

As $\varphi : \{0,1\}^* \to \{0,1\}^*$ is α-expansive, for every $\varepsilon > 0$ there exists some n_0 such that $l(\varphi(\xi|_n)) \geq (\alpha - \varepsilon)n$ for all $n \geq n_0$. Hence, for every $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{K(\hat{\varphi}(\xi)|_n)}{n} \leq \liminf_{n\to\infty} \frac{K(\xi|_n)}{(\alpha-\varepsilon)n}.$$

It follows with Theorem 3.8 that

$$\dim^1_{\mathrm{H}}(\hat{\varphi}(\xi)) = \liminf_{n\to\infty} \frac{K(\hat{\varphi}(\xi)|_n)}{n} \leq \liminf_{n\to\infty} \frac{1}{\alpha}\frac{K(\xi|_n)}{n} = \dim^1_{\mathrm{H}}(\xi).$$

This completes the proof. □

Now suppose f satisfies a Hölder condition from below:

$$\exists \alpha, c > 0\, \forall \xi, \omega \in \{0,1\}^\infty\; c\, d(\xi,\omega)^\alpha \leq d(f(\xi), f(\omega)). \tag{6}$$

Note that this implies that f is injective. Suppose further that f has a recursive monotone representation $\varphi : \{0,1\}^* \to \{0,1\}^*$ with $\hat{\varphi} = f$ that is injective, too. This means that $K(x|\varphi(x), K(\varphi(x))) = O(1)$, since we can always simply scan through all possible strings for a preimage of a given $\varphi(x)$.

Therefore, we get $K(\varphi(x)) = K(x) + O(1)$, and using the lower bound on the length of $\varphi(x)$, we see that, for any ξ,

$$\underline{K}(\varphi(\xi|_n)) \geq \frac{1}{\alpha}\underline{K}(\xi|_n).$$

Combining this observation with Theorem 3.10, the invariance of effective dimension under recursive bi-Lipschitz mappings follows.

Corollary 3.11. *Let $f : \{0,1\}^\infty \to \{0,1\}^\infty$ be a bi-Lipschitz transformation such that there exists a recursive, 1-expansive, injective mapping $\varphi : \{0,1\}^* \to \{0,1\}^*$ with $\hat{\varphi} = f$. Then, for any $X \subseteq \{0,1\}^\infty$,*

$$\dim^1_{\mathrm{H}}(f(X)) = \dim^1_{\mathrm{H}}(X).$$

After having introduced effective dimension, we can now start to determine the dimension of some classes occurring in computability theory such as degrees, spans, etc., especially of those known to have effective measure 0. In particular, we may try to answer the following questions:

- What are interesting examples of classes of non-integral dimension such as the *middle-third Cantor set*
$$C_{1/3} = \left\{ \sum_{i=1}^{\infty} \zeta(i)3^{-i} : \zeta \in \{0,2\}^\infty \right\}$$
(which has dimension $\log 2/\log 3$) in the classical setting?
- Is there a class of effective measure 0 but effective dimension 1?
- Which (nontrivial) classes have effective dimension 0?

4. Some examples of effective dimension

We now present some results on effective Hausdorff dimension. We consider classes arising in the context of computability theory.

4.1. **A small class of maximal dimension.** First, we further employ the Kolmogorov complexity characterization of effective dimension (Theorem 3.8) to get invariance results for other, not necessarily recursive mappings. This will allow us to show that the effective dimension of a degree of a set and its lower span coincide.

Note that the *symmetry of algorithmic information* for prefix complexity says that there exists some constant c such that for any $x, y \in \{0,1\}^*$

$$K(x,y) = K(x) + K(y|x, K(x)) + c \tag{7}$$

A proof of this identity can be found in [11]. Rewriting this in two different ways and replacing y by $\varphi(x)$, where φ maps strings to strings, we get

$$K(\varphi(x)) = K(x) + K(\varphi(x)|x, K(x)) - K(x|\varphi(x), K(\varphi(x))) + c. \tag{8}$$

Next, we have to generalize the notion of a *join* of two sets (sequences). Let Z be an infinite recursive subset of $\mathbb{N}$ with infinite complement. We shall call Z simply a *recursive partition*.

Define the *Z-join* of two sequences $\xi, \omega \in \{0,1\}^\infty$, $\xi \oplus_Z \omega$, to be the unique sequence ζ which satisfies

$$\zeta|_Z = \xi \quad \text{and} \quad \zeta|_{\overline{Z}} = \omega.$$

If $\lim_{n\to\infty} \frac{1}{n}\,|Z \cap \{0,\dots,n-1\}|$ exists, call this number the *density* δ of Z and consider, for given ω, the "insertion mapping" $g : \xi \to \xi \oplus_Z \omega$. This mapping g satisfies a Hölder condition: Define $\varphi : \{0,1\}^* \to \{0,1\}^*$ by

$$\varphi(\xi|_n) = (\xi \oplus_Z \omega)|_{\pi_Z(n)},$$

where $\pi_Z(n)$ denotes the nth element of Z. Note that for each $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that

$$\tfrac{n}{\delta+\varepsilon} - c_1 \le \pi_Z(n) \le \tfrac{n}{\delta-\varepsilon} + c_1$$

for all n. Furthermore, $\hat{\varphi} = g$ and both φ and g are injective. Now we can use (8) to determine the complexity of φ. Obviously, since φ is injective and x is contained in $\varphi(x)$, i.e., can be identified in $\varphi(x)$ since Z is recursive, $K(x|\varphi(x), K(\varphi(x)))$ is bounded by a constant. On the other hand, in order to compute $\varphi(x)$ given x, it suffices to specify the bits that are "inserted" into x (at the $\overline{Z}$-positions). These are at most $\pi_Z(n) - n$ bits, and it follows that, for every $\varepsilon > 0$, $\varepsilon < \delta$,

$$K(\varphi(x)|x, K(x)) \le l(x)\,(\tfrac{1}{\delta-\epsilon} - 1) + O(1).$$

Hence, if $\delta = 1$, we can conclude that, for every $X \subseteq \{0,1\}^\infty$,

$$\dim^1_{\mathrm{H}}(X) = \dim^1_{\mathrm{H}}(\hat{\varphi}(X)).$$

We now use this invariance property to code information into classes of sequences without changing the effective dimension of these classes.

Let r be a standard reducibility in computability theory, i.e., one of 1, m, $btt(1)$, btt, tt, wtt, T. For a set $A \subseteq \mathbb{N}$, let $\deg_r(A)$ and $\mathrm{span}_r(A)$ be its r-degree and lower r-span, respectively.

Theorem 4.1. *For any set $A \subseteq \mathbb{N}$, it holds that*

$$\dim^1_{\mathrm{H}}(\deg_r(A)) = \dim^1_{\mathrm{H}}(\mathrm{span}_r(A))$$

Proof. Let $A \subseteq \mathbb{N}$. Obviously, $\dim^1_{\mathrm{H}}(\deg_r(A)) \le \dim^1_{\mathrm{H}}(\mathrm{span}_r(A))$. To obtain the reverse inequality, choose some recursive partition Z with

$$\lim_{n\to\infty} \tfrac{1}{n}\,|Z \cap \{0,\dots,n-1\}| = 1.$$

Define the mapping $f : \{0,1\}^\infty \to \{0,1\}^\infty$ by $f(\xi) = \xi \oplus_Z A$. (As already mentioned, we identify subsets of the natural numbers with their characteristic sequences.) Then, obviously, $f(\mathrm{span}_r(A)) \subseteq \deg_r(A)$, and by the preceding remarks,

$$\dim^1_{\mathrm{H}}(\mathrm{span}_r(A)) = \dim^1_{\mathrm{H}}(f(\mathrm{span}_r(A))) \le \dim^1_{\mathrm{H}}(\deg_r(A)).$$

Equality follows from $\deg_r(A) \subseteq \mathrm{span}_r(A)$. □

Theorem 4.1 allows us to exhibit an interesting example of an effective nullclass that nevertheless has effective dimension 1.

It is a known fact that the lower tt-span of the halting problem K contains a Martin-Löf random sequence, hence it does not have effective measure 0 $\lambda^1(\mathrm{span}_{tt}(K)) \neq 0$.

On the other hand, K does not tt-reduce to a Martin-Löf random sequence, which implies $\lambda^1(\deg_{tt}(K)) = 0$. (For details on this results refer to TERWIJN [21].) Therefore, we have the following corollary.

Corollary 4.2. $\lambda^1(\mathrm{span}_{tt}(K)) \neq 0$ *but* $\dim^1_{\mathrm{H}}(\deg_{tt}(K)) = 1$.

Note that Corollary 4.2 holds for truth-table reducibility only. The lower btt-span of K is known to effective measure 0, and we can strengthen this result.

Theorem 4.3. $\dim^1_{\mathrm{H}}(\deg_{btt}(K)) = 0$ *and* $\dim^1_{\mathrm{H}}(\mathrm{span}_{btt}(K)) = 0$.

Proof. It is sufficient to prove $\dim^1_{\mathrm{H}}(\mathrm{span}_{btt}(K)) = 0$ as the degree is a subset of the lower span. Using the stability theorem, let ξ be any set btt-reducible to K. There is a constant c such that every $\xi(n)$ depends only on c places of K, and these places can be computed without querying K. Therefore, one can compute for given n the up to cn places which are necessary to compute $\xi|_n$ from a code for n. Furthermore, one can enumerate K at the queried places until all elements have shown up provided one knows how many will eventually do so. These two numbers can be codes with $(2c+2)\log(n)$ many bits and so one has that the overall number of bits needed to compute $\xi|_n$ is in $O(\log(n))$. It follows that $\dim^1_{\mathrm{H}}(\xi) = 0$ and thus $\dim^1_{\mathrm{H}}(\mathrm{span}_{btt}(K)) = 0$. □

4.2. **Another example of a class of dimension** 0**.** KOLMOGOROV has made an easy but fundamental observation.

Theorem 4.4. *Let* $A \subseteq \mathbb{N} \times \{0,1\}^*$ *be recursively enumerable. Suppose* $A_m = \{x : (m,x) \in A\}$ *is finite. Then, for some constant c and for all* $x \in A_m$,

$$C(x|m) \leq \log|A_m| + c.$$

Here C is the 'plain' (non-prefix) version of Kolmogorov complexity. As plain complexity and prefix complexity are asymptotically equal, we can immediately deduce the following *covering principle*:

Proposition 4.5. *Suppose for some* $\xi \in \{0,1\}^\infty$ *there is an r.e. set* $A \subseteq \{0,1\}^*$ *such that for infinitely many n it holds that* $\xi|_n \in A \cap \{0,1\}^n$. *Then*

$$\dim^1_{\mathrm{H}}(\xi) \leq \liminf_{n\to\infty} \frac{\log|A_n|}{n}.$$

(Compare this with Theorem 3.8.) We give an application of this principle:

A set $A \subseteq \mathbb{N}$ is *semirecursive*, if there is a recursive function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

(I) $f(m,n) \in \{m,n\}$ for all $m,n \in \mathbb{N}$,

(II) $m \in A \vee n \in A$ implies $f(m,n) \in A$.

Semirecursive sets were introduced by JOCKUSCH [10] and have been used to give a structural solution to *Post's Problem* (for details refer to ODIFREDDI's monograph [17]). Note that there are semirecursive sets which are not recursively enumerable (and hence not recursive). In fact, every *tt*-degree contains a semirecursive set (JOCKUSCH). Nevertheless, from the point of view of effective dimension, semirecursive sets are not very complex.

Theorem 4.6. *The class of all semirecursive sets has effective dimension* 0.

Proof. It suffices to show that every semirecursive set (that is, its characteristic sequence) has effective dimension 0.

Let ξ be the characteristic sequence of a semirecursive set. Note that, for any two numbers m, n, we can recursively exclude one of four possible values of the two-bit string $\xi(m)\xi(n)$. This observation was used by BEIGEL, KUMMER and STEPHAN [2], who studied semirecursive sets in the broader context of *frequency computability*, to prove the following lemma.

Lemma 4.7. *If $\xi \in \{0,1\}^\infty$ is the characteristic sequence of a semirecursive set, then there exists a recursive set $C \subseteq \{0,1\}^*$ such that, for any n, $|C \cap \{0,1\}^n| \leq n+1$ and $\xi|_n \in C \cap \{0,1\}^n$.*

The proof of the lemma, which occurs in [2] in a much more general form, uses the SAUER-PERLES-SHELAH-Lemma from extremal combinatorics. Lemma 4.7 allows us to immediately deduce that $\dim^1_{\mathrm{H}}(\xi) = 0$, because of Proposition 4.5. □

4.3. **Lower spans of non-integral dimension.** Of course, when developing a notion of effective Hausdorff dimension, a central interest lies in exhibiting classes of non-integral dimension. In the Cantor space, the most important example was given by BESICOVITCH [3] and EGGLESTON [7], who showed that the class of sequences in which ones occur with a given limiting frequency β has Hausdorff dimension $H(\beta)$, where H denotes the binary entropy function. LUTZ [13] has shown that these results carry over to the effective case. In this paper we further extend this to a special family of lower spans. For this purpose we have to introduce (generalized) Bernoulli measures on the Cantor space.

Let $p_0, p_1, p_2, \ldots$ be a sequence of real numbers such that $0 \leq p_i \leq 1$ for all i. This sequence induces a measure on $\{0,1\}^\infty$ in the following way: Let C_w, $|w| = n$, be an open cylinder. For $i \geq 0$, let $\mu_i(1) = p_i$, $\mu_i(0) = 1 - p_i$ and

$$\mu(C_w) = \prod_{i=0}^{n-1} \mu_i(w(i)) \tag{9}$$

The resulting measure (as can be easily checked) is a probability measure on $\{0,1\}^\infty$, called a *generalized Bernoulli measure*. It is a *Bernoulli measure*, if all the p_i are identical, i.e., $p_i = p$ for all i.

The (generalized) Bernoulli measures can be seen as an infinite product of measures on $\{0,1\}$, endowed with the obvious σ-algebra and a probability measure given by μ_i as defined above. For $p_i = 1/2$ for all $i \geq 0$ we get the usual Lebesgue measure λ on $\{0,1\}^\infty$.

Bernoulli measures can be effectivized in the usual way, even as Hausdorff measures.

Definition 4.8. Let μ be a generalized Bernoulli measure and $s \geq 0$. A class $X \subseteq \{0,1\}^\infty$ has *effective s-dimensional μ-measure* 0, $\mu^{1,s}(X) = 0$, if there exists a recursively enumerable set C such that

$$\sum_{w \in C} \mu(C_w)^s < \infty \quad \text{and} \quad (\forall \xi \in X)\,(\exists^\infty w \in C)\,[w \prec \xi]. \tag{10}$$

It is possible to define a concept of effective μ-dimension as well (the first author has done so), but here we restrict our attention to *μ-random sequences*.

Definition 4.9. Let μ be a generalized Bernoulli measure. A sequence $\xi \in \{0,1\}^\infty$ is 1-*μ-random*, if it does not have effective 1-dimensional μ-measure 0, i.e., $\mu^{1,1}(\{\xi\}) \neq 0$.

1-μ-random sequences can be regarded as typical outcomes of an infinite sequence of coin tosses, where in each round the bias of the coin is chosen according to μ_i. The notion of a 1-μ-random sequence with respect to a generalized Bernoulli measure was already considered by MARTIN-LÖF, when he gave his definition of algorithmic randomness.

Now we turn to a special class of generalized Bernoulli measures. Let μ be such a measure with bias sequence $p_0, p_1, p_2, \ldots$ converging to a real number $\beta \in (0,1)$. The law of large numbers implies that μ-almost every sequence in $\{0,1\}^\infty$ must have limiting frequency β, that is

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \xi(i)}{n} = \beta \ \mu\text{-almost surely}. \tag{11}$$

In particular, every 1-μ-random sequence satisfies (11).

Building on results by EGGLESTON [7], LUTZ [13] showed the following:

Theorem 4.10. *Let μ be a generalized Bernoulli measure generated by a computable bias sequence $p_0, p_1, p_2, \ldots$ of rational numbers converging to a real number $\beta \in (0,1)$. Then, for every 1-μ-random sequence ξ,*

$$\dim^1_{\mathrm{H}}(\xi) = H(\beta),$$

where $H(\beta) = -[\beta \log \beta + (1-\beta)\log(1-\beta)]$ is the binary entropy function.

Theorem 4.10 is also a straightforward consequence of the characterization of 1-μ-random sequences via Kolmogorov complexity (using sequential tests, see [11]) due to LEVIN and SCHNORR, together with the law of large numbers. Employing this characterization, one can easily extend this result as follows (see [1] for the somewhat more complicated resource-bounded version).

Theorem 4.11. *Let μ be as in Theorem 4.10. Then, for every 1-μ-random sequence ξ,*

$$\dim^1_{\mathrm{H}}(\mathrm{span}_m(\xi)) = H(\beta).$$

Note that Theorem 4.11 implies the existence of lower m-spans of arbitrary Δ^0_2-computable dimension (since H is a continuous surjective function on $[0,1]$).

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