Solomonoff Induction Violates Nicod's Criterion

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Outline

Motivation

What does this green apple tell you about black ravens?

The Paradox of Confirmation

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Paradox?

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Solomonoff Induction

Let U be a universal monotone Turing machine.

Solomonoff's universal prior [Sol64]:

$$M(x) \;:=\; \sum_{p:\, U(p) = x\ldots} 2^{-|p|}$$

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Solomonoff normalization: $M_{\mathrm{norm}}(\epsilon) := 1$ and

$$M_{\mathrm{norm}}(xa) := M_{\mathrm{norm}}(x) \frac{M(xa)}{\sum_{b \in \mathcal{X}} M(xb)}$$

M_{norm} is a measure (probability distribution on $\mathcal{X}^\infty$).

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- M *merges* with any computable measure μ [BD62]:

$$\sup_{H \text{ measurable}} M(H \mid x_{<t}) - \mu(H \mid x_{<t}) \to 0 \ \mu\text{-a.s. as } t \to \infty$$

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$\Longrightarrow$ M is really good at induction!

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Confirmation and disconfirmation:

$$\mu(H) = 0 \implies \exists t.\ M(H \mid x_{<t}) = 0 \ \mu\text{-a.s.}$$
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Question: Does a black raven *confirm* H:

$$M(H \mid x_{<t}) < M(H \mid x_{<t}BR)?$$

Solomonoff Induction and Nicod's Criterion

Theorem (Counterfactual Black Raven Disconfirms H)

Let $x_{1:\infty} \in H \subset \mathcal{X}^\infty$ *be computable and* $x_t \neq BR$ *infinitely often.*
$\Longrightarrow \exists t \in \mathbb{N}$ *(with* $x_t \neq BR$*) s.t.* $M(H \mid x_{<t}BR) < M(H \mid x_{<t})$

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Let $x_{1:\infty} \in H$ *be computable.*
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Theorem (Disconfirmation Infinitely Often for M_{norm})

There is an (incomputable) $x_{1:\infty} \in H$ *s.t.*
$M_{\mathrm{norm}}(H \mid x_{1:t}) < M_{\mathrm{norm}}(H \mid x_{<t})$ *infinitely often.*

Proof

$M(\cdot \cap \cdot)$	H	H^c
$\bigcup_{a \neq x_t} \Gamma_{x_{<t}a}$	A	B
$\Gamma_{x_{1:t}}$	C	D
$\{x_{<t}\}$	E	0

Lemma

(i) $0 < A, B, C, D, E < 1$

(ii) $A + B \overset{\times}{\leq} 2^{-K(t)}$

(iii) $A, B \overset{\times}{\geq} 2^{-K(t)}$

(iv) $C \overset{\times}{\geq} 1$

(v) $D \overset{\times}{\geq} 2^{-m(t)}$

(vi) $D \to 0$ *as* $t \to \infty$

(vii) $E \to 0$ *as* $t \to \infty$

$M(H \mid x_{1:t}) > M(H \mid x_{<t})$ if and only if $AD + DE < BC$

Outline

Resolving the Paradox of Confirmation I

Solution: Reject Nicod's criterion!

Not all black ravens confirm H.

Resolving the Paradox of Confirmation II

E. T. Jaynes [Jay03, p. 144]:

> *In the literature there are perhaps 100 ‘paradoxes’ and controversies which are like this, in that they arise from faulty intuition rather than faulty mathematics. Someone asserts a general principle that seems to him intuitively right. Then, when probability analysis reveals the error, instead of taking this opportunity to educate his intuition, he reacts by rejecting the probability analysis.*

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