# New Directions in Randomness

Jason Rute

Pennsylvania State University

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Slides available at
www.personal.psu.edu/jmr71/

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# Introduction

# The goals

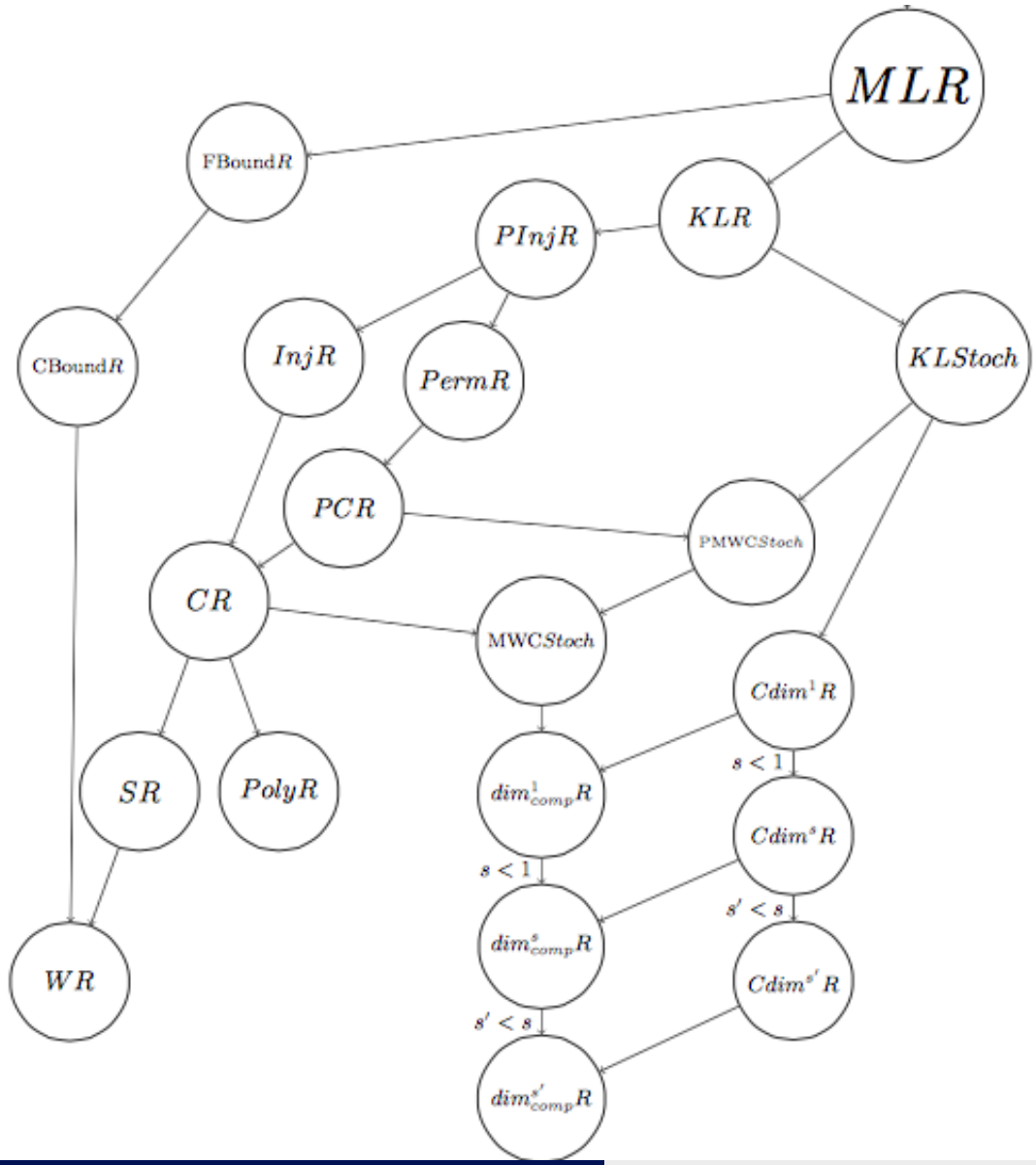
- To make you think about randomness in a new way.
- What is a randomness notion?
- What is a natural randomness notion?
- Can randomness be studied as a theory? Like the theory of groups?
- Can we axiomatize algorithmic randomness?

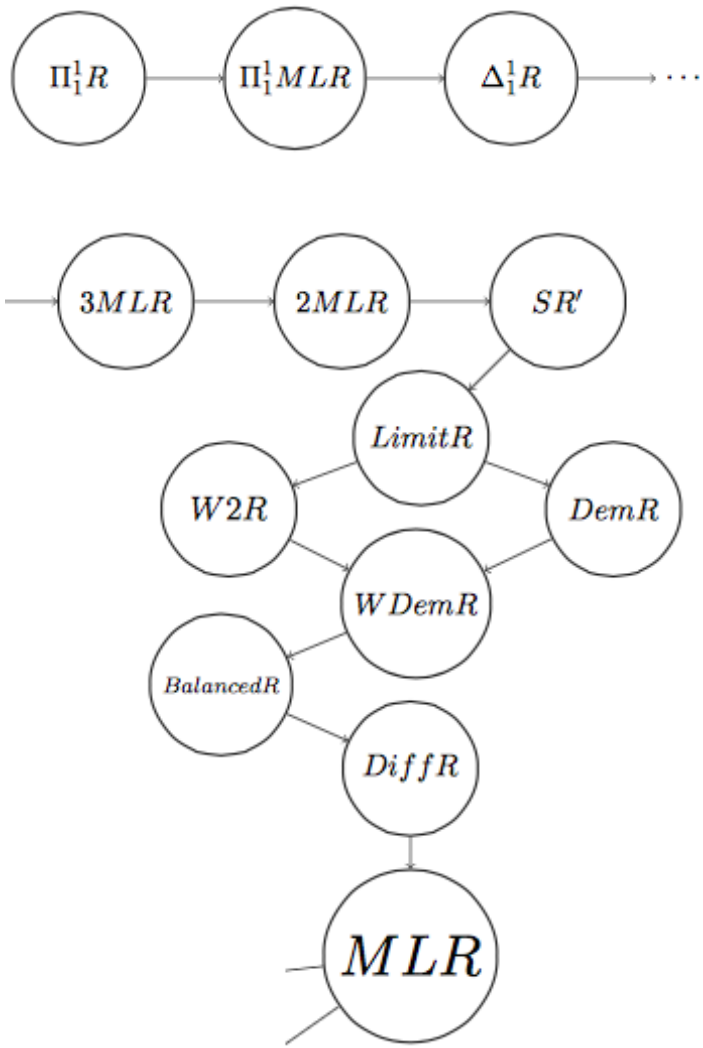
# Organizing the randomness zoo

# The Heidelberg zoo

# The randomness zoo

Antoine Taveneaux

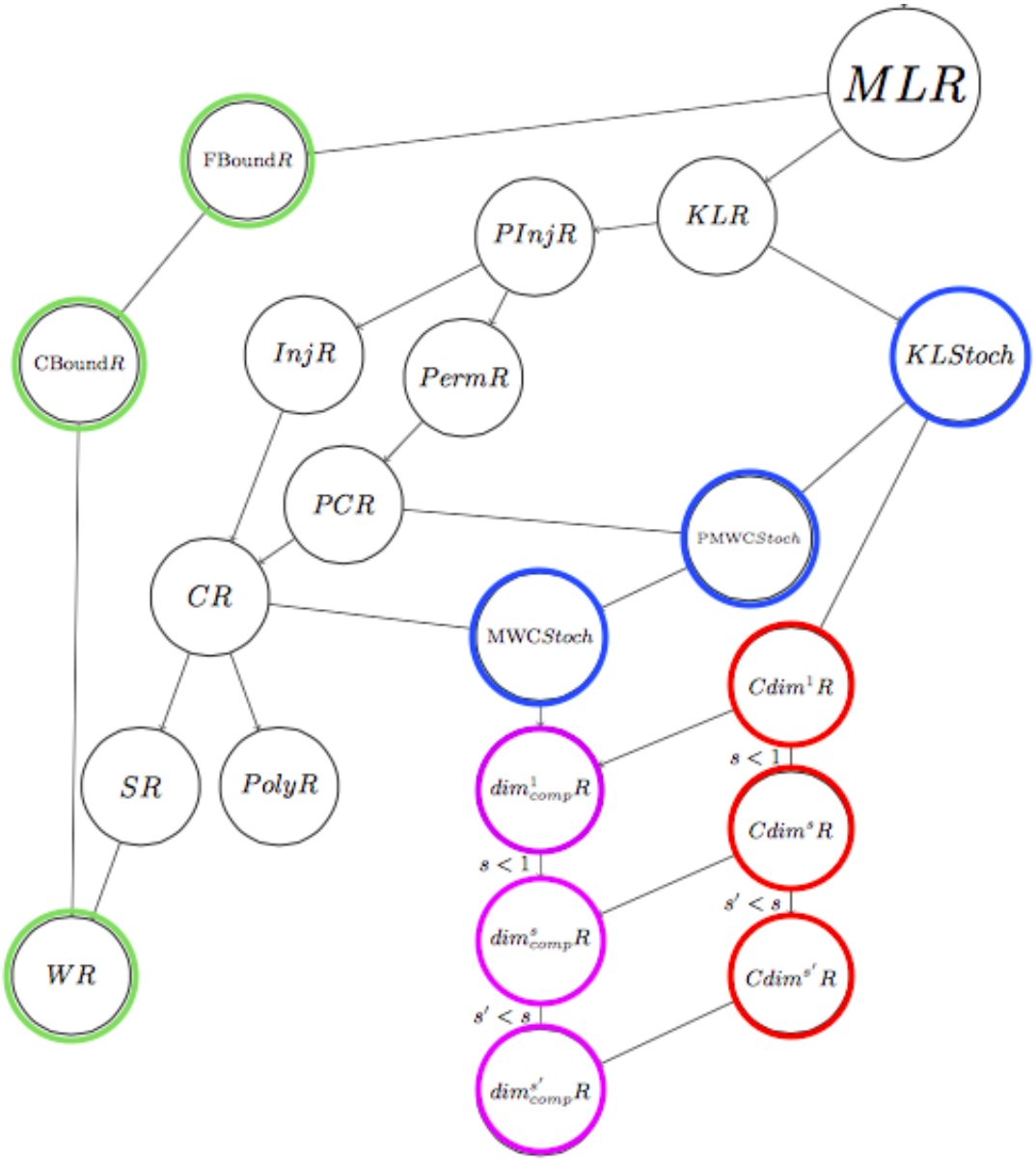


# Organizing the randomness zoo

## Step 1: Organize by σ-ideals

# Some randomness notions are not like the others




- Kurtz-like (green)
- Stochastic (blue)
- Partial randomness (purple/red)

- This can largely be explained via σ-ideals.

# σ-ideals

- A **σ-ideal** is a collection of sets closed downward and under countable unions.
- Each σ-ideal $\mathcal{I}$ provides a notion of "small set" or "null set".
- Examples:
  - meager sets
  - null sets
  - sets of Hausdorff dimension $\leqslant s$ (for a fixed $0 \leqslant s \leqslant 1$).
- Every "randomness" notion is associated with a σ-ideal $\mathcal{I}$.

# Example: σ-ideals of Kurtz randomness

- $x \in 2^{\mathbb{N}}$ is **Kurtz random** (or **weak random**) if $x$ is not in any $\Pi^0_1$ null set.
- Common complaint: "Kurtz randomness is really a genericity notion."

- Let $\mathsf{Kurtz}^A$ be the set of $A$-Kurtz random sequences for the oracle $A$.
- Let $\mathcal{I}_{\mathsf{Kurtz}}$ be the σ-ideal of subsets of $2^{\mathbb{N}} \setminus \mathsf{Kurtz}^A$ for some $A$.
- $\mathcal{I}_{\mathsf{Kurtz}}$ is the exactly the σ-ideal of subsets of $F_\sigma$ (i.e. $\mathbf{\Sigma}^0_2$) null sets.
- These are the null sets associated with Riemann integrable functions, a.e. continuous functions, and Jordan-Peano measurable sets.
- $\mathcal{I}_{\mathsf{Kurtz}}$ is a sub-σ-ideal of both the ideals of meager sets and the ideal of null sets.

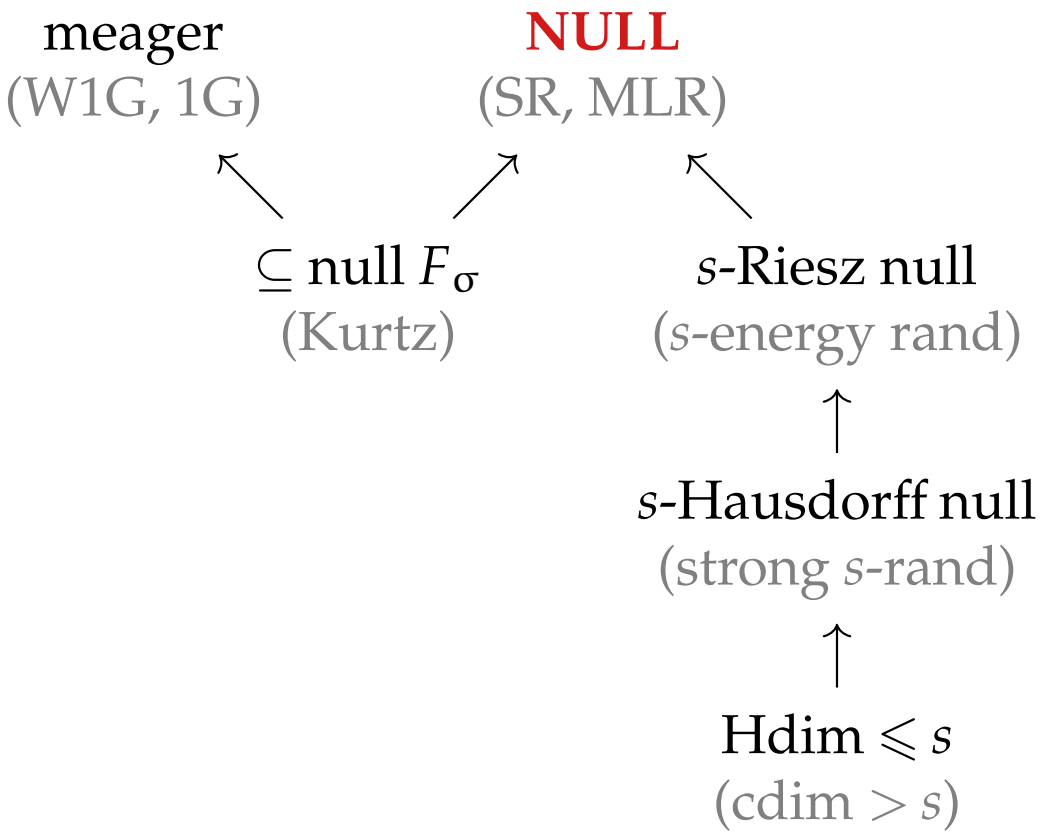
- Kurtz randomness is both a genericity notion and a randomness notion.

# σ-ideals and their "randomness notions"

| σ-Ideal | Randomness (Genericity) notions |
|---|---|
| Meager | weakly 1-generic, 1-generic |
| Subsets of $F_\sigma$-null | Kurtz, finite bounded, Kurtz$^{\emptyset'}$ |
| **(Lebesgue) null** | **Sch, CR, ML, W2R, 2R, etc.** |
| **μ-null** | **μ-Sch, μ-CR, μ-ML, μ-W2R, μ-2R, etc.** |
| Hausdorff dimension $\leqslant s$ | Sch-dim $> s$, cdim $> s$ |
| Null $s$-dim. Hausdorff measure | strong $s$-randomness: $KM(x \upharpoonright n) \geqslant^+ sn$ |
| Null $s$-dim. Riesz capacity | $s$-energy randomness: $\sum_n 2^{sn - KM(x \upharpoonright n)} < \infty$ |

- It is not clear what the σ-ideals are for
  - the stochasticity notions
  - constructive dimension = 1
  - (weak) $s$-randomness
  - UD randomness
- However, they are clearly not the σ-ideal of Lebesgue null sets.

# σ-ideal zoo




- From here on, we will focus on the σ-ideal of Lebesgue (or μ-) null sets.

# Organizing the randomness zoo

## Step 2: Organize by computability

# True randomness vs. algorithmic randomness

- $x$ is **truly random** if $x$ avoids every null set.
- Except for a pesky problem...

- Our "solution" is to consider **algorithmic** null sets.
- However, what type of algorithmic?

# Levels of computability in algorithmic randomness

- Poly-time randomness notions
  - Poly-time Schnorr random
  - Poly-time random
  - ...
  - Computable randomness notions
    - Schnorr random
    - Computably random
    - Martin-Löf random
    - Weak 2-random
    - 2-random
    - ...
    - Higher randomness notions
      - $\Delta^1_1$ random
      - $\Pi^1_1$ MLR random
      - $\Pi^1_1$ random
      - ...
- Forcing randomness notions
  - Solovay genericity
  - ...
  - "Pointless" randomness notions
    - True randomness
- **From now on, we will just work at the computable level.**

# Organizing the randomness zoo

## Step 3: Mark the minimal sufficient randomness notion in each computability level

# Schnorr randomness is sufficient

- A **μ-Schnorr test** is a computable sequence of $\Sigma^0_1$ sets such that $\mu(U_n) \leqslant 2^{-n}$ and $\mu(U_n)$ is computable in $n$.
- $x$ is **μ-Schnorr random** if $x \notin \bigcap_n U_n$ for any μ-Schnorr test.

- Schnorr randomness is closely connected to constructive mathematics.
- See the slides for my VAI 2015 talk (available on my webpage).
- Schnorr null sets where first called "null sets in the sense of Brouwer."
- Constructively provable a.e. theorems are true for Schnorr randomness.

# Schnorr randomness is minimally sufficient

- Schnorr randomness is the minimal randomness notion for working with **computable measurable objects.**

## Definition

A function $f\colon 2^{\mathbb{N}} \to \mathbb{R}$ is **$L^1$-computable** if there is a computable sequence of rational step functions $f_n$ such that

$$\|f_n - f\|_1 = \int |f_n - f| \, d\mu \leqslant 2^{-n}.$$

- Only on Schnorr randoms is the convergence of $f_n(x)$ guaranteed.
- Moreover, if the computable sequence $g_n$ also converges rapidly to $f$ in $L^1$, then $\lim_n g_n(x) = \lim_n f_n(x)$ for all Schnorr randoms $x$.

- This is one of many such similar examples.

# Other computability notions

- There is no obvious reason why these ideas cannot be extended to lower and higher computability notions.

## Conjectures

1. **Poly-time Schnorr randomness** is the minimal sufficient randomness notion with respect to **poly-time computability**.
2. **Higher Schnorr randomness** (i.e. $\Delta^1_1$ **randomness**) is the minimal sufficient randomness notion with respect to **higher computability**.

- These conjectures extend to basically every idea in this talk.

# Organizing the randomness zoo

## Step 4:
## Separate the wheat from the chaff,
## the sheep from the goats,
## the good randomness notions from the bad

# Work with many randomness notions at once

- Why prove a theorem for one randomness notion when you can prove it for all of them?

- For example, the theorem

  *Schnorr randomness satisfies the strong law of large numbers.*

  holds for all stronger randomness notions (CR, MLR, W2R, 2R, etc.).

- However, many theorems of randomness are not of this form.
- For example,

  *Schnorr randomness is closed under computable permutations of bits.*

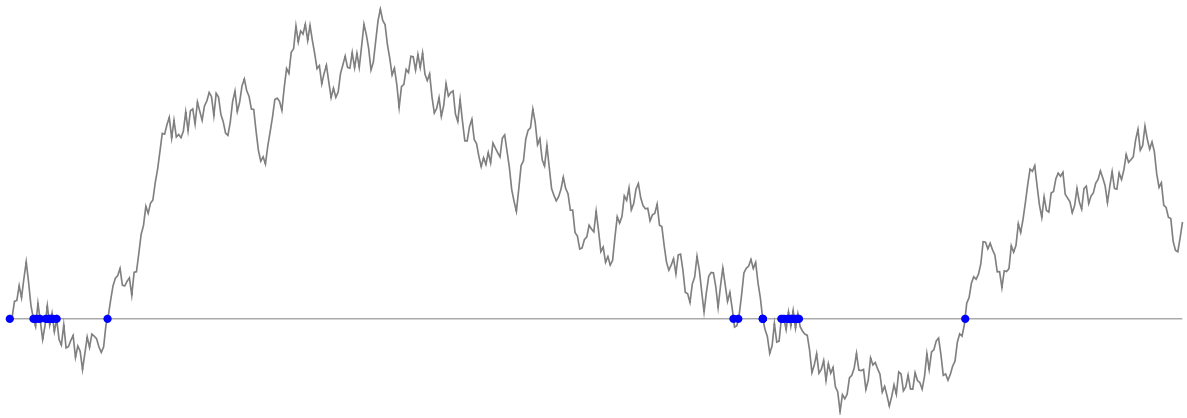
  is not satisfied by partial computable randomness (PCR) even though PCR is stronger than Schnorr randomness.

# Developing a framework of randomness notions

- The rest of this talk is devoted to developing a **system of axioms** which are sufficient for **working with randomness in practice**.
- The randomness notions satisfying these axioms are the natural ones.
- The unnatural ones should be demoted to footnotes in our zoo.

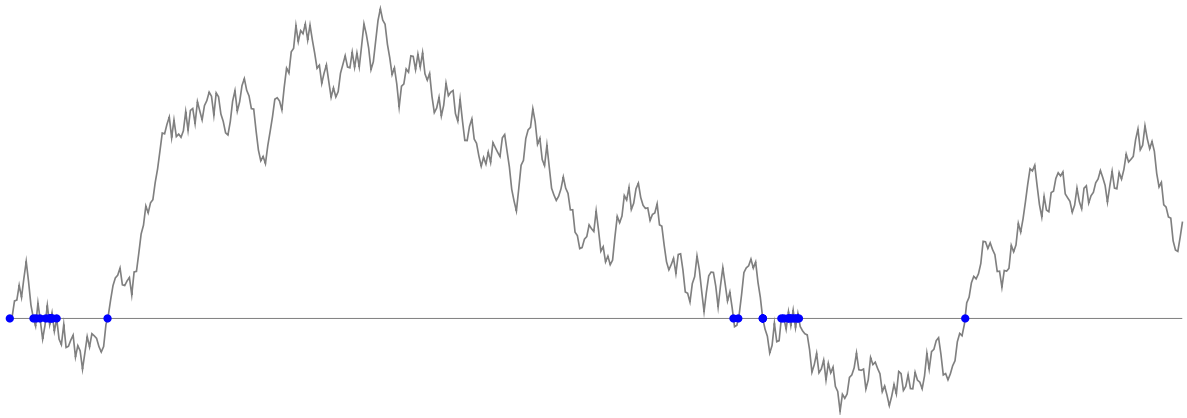
# Properties desired of an algorithmic randomness notion

# A very informal guiding principle



- A natural randomness notion should be sufficient for working constructively with Brownian motion

# Extendable to other spaces



- Brownian motion is given by the Wiener measure on $C[0,1]$ or $C[0,\infty)$.

## Generalization

Randomness should generalize to all computable probability spaces $(\Omega, \mathbf{P})$.

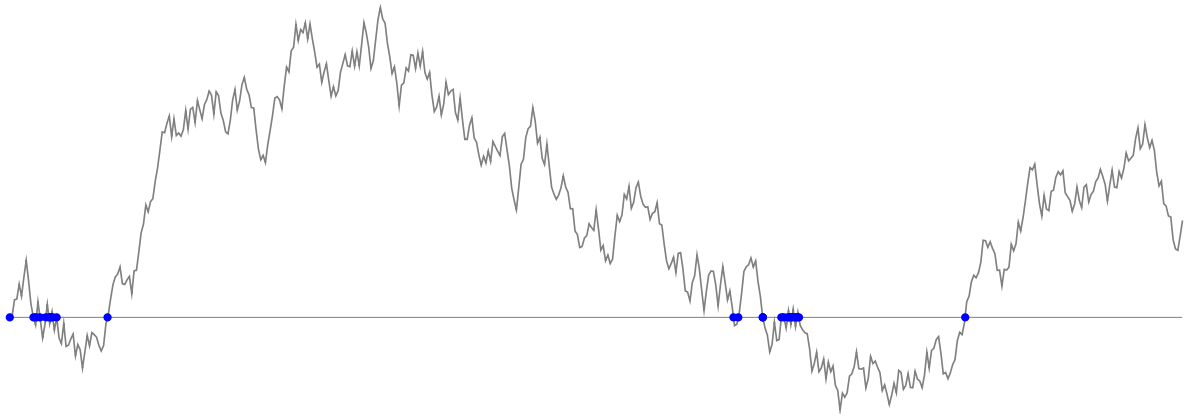
# Extendable to other spaces

- Schnorr randomness, Martin-Löf randomness, weak-n-randomness, $n$-randomness are all naturally extendable to other spaces.
- Computable randomness also has a consistent extension to other probability spaces (R.).
  - A **measure bounded integral test** on $(X, \mu)$ is a lowersemicomputable function $t\colon X \to [0, \infty]$ and a computable measure $\nu$ such that

$$\int_A t(x)\, d\mu(x) \leqslant \nu(A) \quad (A \subseteq X \text{ measurable}).$$

  - $x \in X$ is **$\mu$-computably random** if $t(x) < \infty$ for all measure bounded integral tests $t$.
- For some of the more combinatorial randomness notions (e.g. partial computable randomness or Kolmogorov-Loveland randomness) it is not so clear.

# Invariant under isomorphisms



- Brownian motion can be transformed via a number of isomorphisms.
- For example, if $B(t)$ is a BM, then the following are BMs:

$$-B(t) \qquad \text{and} \qquad tB(1/t).$$

- Moreover, all the standard constructions of BM are isomorphisms between other probability spaces and the Wiener measure.

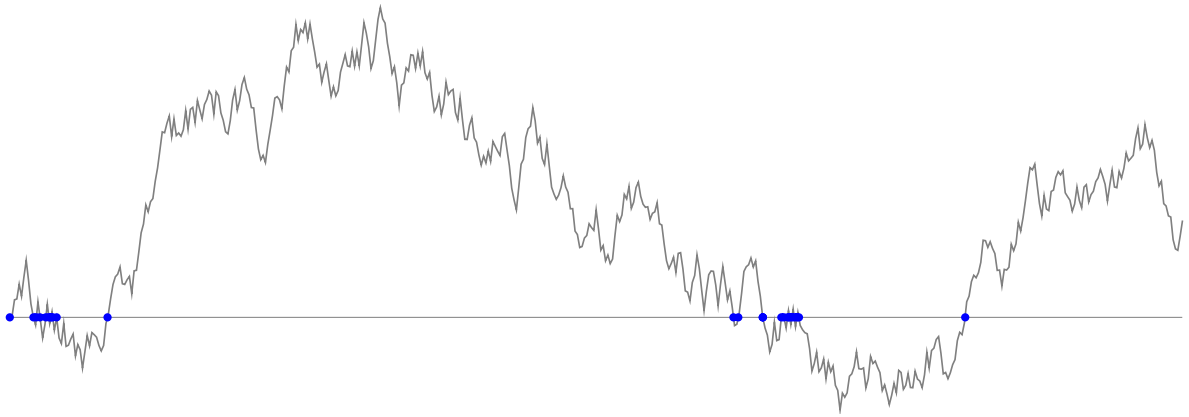
## Preservation under isomorphisms

If $I\colon (\Omega_1, \mathbf{P}_1) \simeq (\Omega_2, \mathbf{P}_2)$ is an effectively measurable isomorphism, then $\omega$ is $\mathbf{P}_1$-random if and only if $I(\omega)$ is $\mathbf{P}_2$-random.

# Invariant under isomorphisms

- Schnorr randomness, Martin-Löf randomness, weak-n-randomness, $n$-randomness are all invariant under isomorphisms.
- Computable randomness is also invariant under isomorphisms (R.).
- Partial computable randomness is not invariant under permutations of bits.

# Randomness preservation



- The probability distribution of $B(1)$ is the Gaussian measure on $\mathbb{R}$.
- In other words, the Gaussian measure is the push-forward of the Wiener measure along the map $B \mapsto B(1)$.

## Preservation of randomness

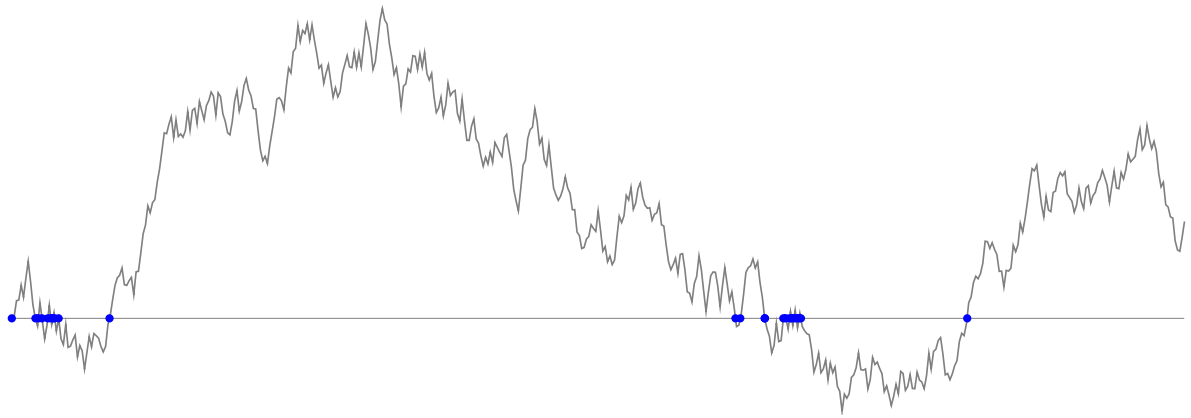
Assume $T\colon (\Omega, \mathbf{P}) \to (X, \mathbf{P}_T)$ is an effectively measurable map. If $\omega$ is $\mathbf{P}$-random, then $T(\omega)$ is $\mathbf{P}_T$-random. (Here $\mathbf{P}_T$ is the pushforward measure of $\mathbf{P}$ along $T$.)

# Randomness preservation

- Schnorr randomness, Martin-Löf randomness, weak-n-randomness, $n$-randomness all satisfy randomness preservation.
- Computable randomness does not (Bienvenu/Porter; R.).

- Although, I will have more to say about this in a bit...

# Equivalent measures share randoms



- The Gaussian measure and the Lebesgue measure on $\mathbb{R}$ are equivalent measures, i.e. they have the same null sets.

**Equivalent measures share randoms**

"Effectively equivalent" measures have the same randoms.

# Equivalent measures share randoms

- This property can be stated with the following two properties.

## Equivalent measures share randoms

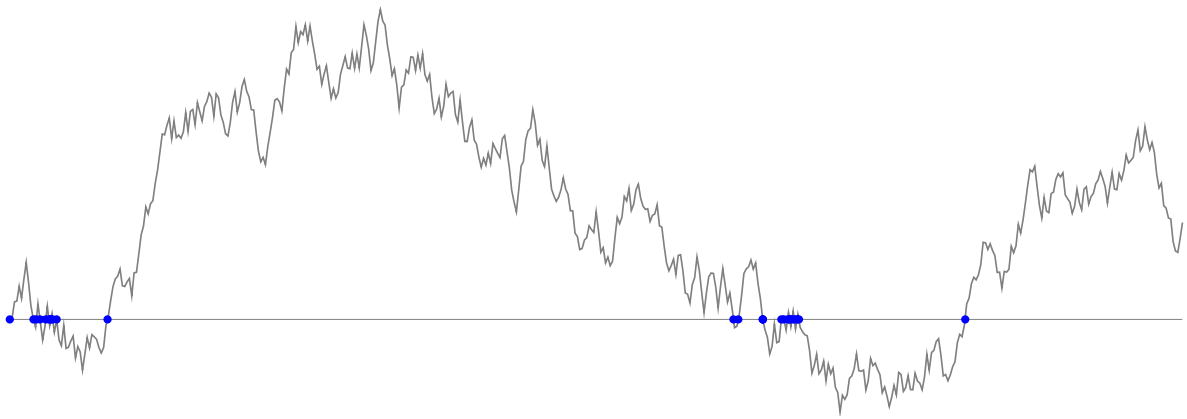
1. If $x$ is $\mu$-random and $\mu \leqslant c\nu$ for some constant $c$, then $x$ is $\nu$-random.
2. Assume $\mu \ll \nu$ with an $L_1(\nu)$-computable density $f = \frac{d\mu}{d\nu}$, that is

$$\mu(A) = \int_A f \, d\nu \quad (A \subseteq X).$$

Then, $x$ is $\mu$-random iff both $x$ is $\nu$-random and $f(x) > 0$

- The standard randomness notions satisfy both of these:
  - SR, CR, MLR, n-random, weak $n$-random

# No randomness from nothing



- Again consider that a Gaussian distribution can be found from a Brownian distribution.

**No randomness from nothing (a.k.a no randomness ex nihilo)**

Assume $T\colon (\Omega, \mathbf{P}) \to (X, \mathbf{P}_T)$ is an effectively measurable map. If $x$ is $\mathbf{P}_T$-random, then there is a $\mathbf{P}$-random $\omega$ such that $x = T(\omega)$.

# No randomness from nothing

- No-randomness-from-nothing holds for Martin-Löf randomness, $n$-randomness, weak 2-randomness, difference randomness.

**Theorem (R.)**

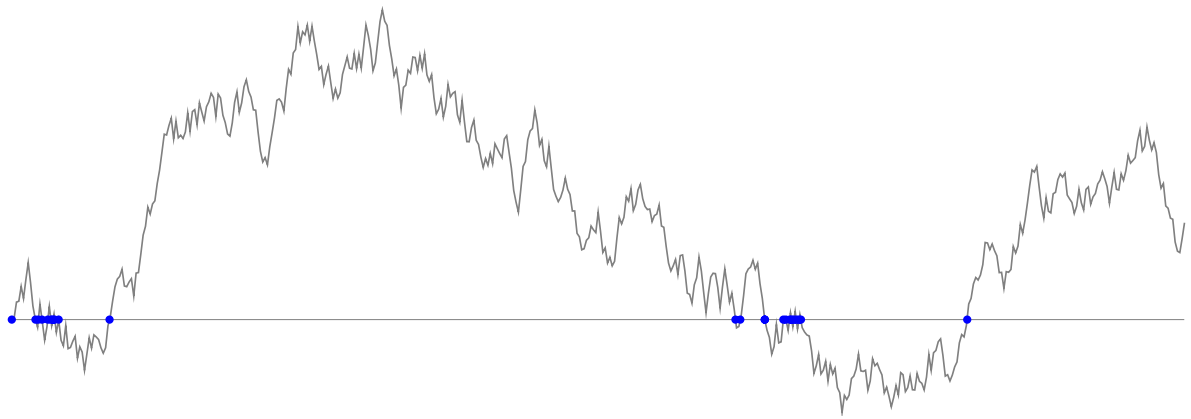
- No-randomness-from-nothing holds for computable randomness.
- However, it does not hold for Schnorr randomness:
- If $x$ is not CR, then there is a measure-preserving almost-everywhere computable map $T$ such that the preimage of $x$ under $T$ is empty.

**Theorem (R.)**

- Martin-Löf randomness is the weakest randomness notion satisfying both no-randomness-from-nothing and randomness preservation.

- It is interesting (but not damning!) that NRFN fails for SR.

# Van Lambalgen and combining measures



- A Brownian motion on $[0,1]$ can be **constructed** by "gluing together" two independent BM on $[0,1/2]$.
- And vice versa, a Brownian motion on $[0,1]$ can be **decomposed** into two independent BM on $[0,1/2]$.

## Van Lambalgen's theorem

$(\omega_1, \omega_2)$ is $\mathbf{P}_1 \times \mathbf{P}_2$-random iff $\omega_1$ is $\mathbf{P}_1$-random and $\omega_2$ is $\mathbf{P}_2$-random independently of $\omega_1$.

# Independence

## Van Lambalgen's theorem

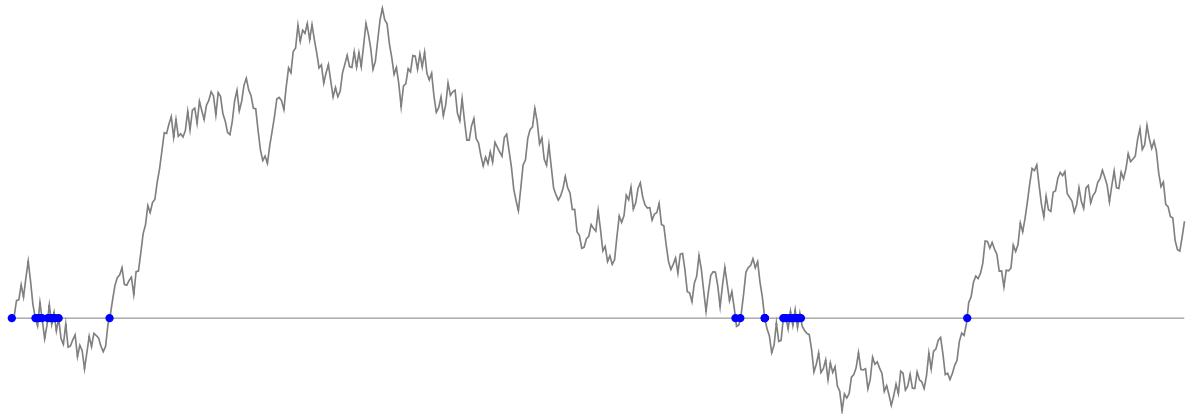
$(\omega_1, \omega_2)$ is $\mathbf{P}_1 \times \mathbf{P}_2$-random iff $\omega_1$ is $\mathbf{P}_1$-random and $\omega_2$ is $\mathbf{P}_2$-random **independently of** $\omega_1$.

- "Independent" is often taken one of two ways:
  - $\omega$ is $\mathbf{P}$-random **relative** to $A$ means there is no test $T^A$ computable from $A$ that derandomizes $\omega$.
  - $\omega$ is $\mathbf{P}$-random **uniformly relative** to $A$ means there is no computably indexed family of tests $\{T^B\}$, one test for each oracle $B$, such that $T^A$ derandomizes $\omega$.
- For Martin-Löf and $n$-randomness, relative and uniformly relative are the same.
- (Others have suggested that "independent" should mean whatever makes van Lambalgen's theorem holds.)

# Van Lambalgen's theorem

- Martin-Löf randomness and $n$-randomness satisfy van Lambalgen's theorem with both **uniform relativization** and **relativization** (because they are the same!).
- The following satisfy van Lambalgen's theorem for **uniform relativization**:
  - Schnorr randomness (Miyabe; Miyabe and R.)
  - Demuth randomness (Diamondstone, Greenberg, Turetsky)
- For computable randomness
  - One direction is true for **uniform relativization** (Miyabe).
  - The other direction fails for both types of relativization (Bauwens, last week!)
- For other types of randomness, the details are not fully worked out.

# Van Lambalgen's theorem gives other results



- Notice that one can construct a Brownian motion with two steps:
  1. Choose a value $a$ at $t = 1$ from a Gaussian distribution.
  2. Connect $(0,0)$ to $(1,a)$ via a **Brownian bridge ending at** $a$
- The distribution in the second step is computable uniformly from the chosen $a$.
- Using this idea we can, in many cases, recover randomness preservation for computable randomness and no-randomness-from-nothing for Schnorr randomness.

# Generalized van Lambalgen's theorem

- Let $(\Omega_1, \mathbf{P}_1)$ be a computable probability measure.
- Let $\mathbf{P}(\cdot \mid \omega)$ be a **computable kernel**, that is a family of probability measures on the space $\Omega_2$ such that the map $\omega \mapsto \mathbf{P}(\cdot|\omega)$ is effectively measurable.
- Combine $\mathbf{P}_1$ and $\mathbf{P}(\cdot \mid \omega)$ into one probability space $(\Omega_1 \times \Omega_2, \mathbf{P})$ via

$$\mathbf{P}(A \times B) = \int_A \mathbf{P}(B \mid \omega_1)\, d\mathbf{P}_1(\omega_1).$$

## Generalized van Lambalgen's theorem

$(\omega_1, \omega_2)$ is $\mathbf{P}$-random iff $\omega_1$ is $\mathbf{P}_1$-random and $\omega_2$ is **$\mathbf{P}(\cdot \mid \omega_1)$-random** independently of $\omega_1$.

- Besides interpreting "independently", we also have to figure out what "$\mathbf{P}(\cdot \mid \omega_1)$-random" means since this measure may not be computable
- It could mean using $\mathbf{P}(\cdot \mid \omega_1)$ as an oracle.
- It could mean using $\mathbf{P}(\cdot \mid \omega_1)$ uniformly as an oracle.

# Generalized van Lambalgen's theorem

- Generalized van Lambalgen's theorem holds for
  - Martin-Löf randomness (Takahashi)
  - Schnorr randomness (R., using uniform relativization)

# Van Lambalgen's theorem for maps

- Assume $T\colon (\Omega, \mathbf{P}) \to (X, \mathbf{P}_T)$ is an effectively measurable map.
- Assume the conditional probability $x \mapsto \mathbf{P}(\cdot \mid T = x)$ is effectively measurable as a map from $(X, \mathbf{P}_T)$ to measures.

## van Lambalgen's theorem for maps

$$\begin{pmatrix} \omega \text{ is } \mathbf{P}\text{-random} \\ \& \quad x = T(\omega) \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \text{ is } \mathbf{P}_T\text{-random} \quad \& \\ \omega \text{ is } \mathbf{P}(\,\cdot \mid T = x)\text{-random independent of } x \end{pmatrix}$$

- The $\Rightarrow$ direction is a stronger version of randomness preservation.
- The $\Leftarrow$ version is a stronger version of no-randomness-from-nothing.
- It also lets one prove that if $P \ll Q$ with an $L^1$-computable density function $f$, then $x$ is $P$-random if and only if $x$ is $Q$-random and $f(x) > 0$.

# Proposed axioms of randomness

# Tentative randomness axioms

- $\langle x, \mu, a \rangle \in \mathcal{R}$ means $x$ is $\mu$-random independent of $a$.

- Axiom 1: For all $\mu$ and $a$, $\mu\{x : \langle x, \mu, a \rangle \in \mathcal{R}\} = 1$.
- Axiom 2: If $\langle x, \mu, a \rangle \in \mathcal{R}$, then $x$ is $\mu$-Schnorr random uniformly relativized to $a$.
- Axiom 3: If $b$ is computable uniformly in $(a, \mu)$, then $\langle x, \mu, a \rangle \in \mathcal{R}$ implies $\langle x, \mu, b \rangle \in \mathcal{R}$.
- Axiom 4: If $\mu$ is computable uniformly in $a$, $T \colon \Omega \to \Omega$ is $\mu$-effectively measurable uniformly in $a$, and $y \mapsto \mu(\cdot \mid T = y)$ is $\mu_T$-effectively measurable uniformly in $a$, then

$$\begin{pmatrix} \langle x, \mu, a \rangle \in \mathcal{R} \\ \text{and } y = T(x) \end{pmatrix} \Leftrightarrow \begin{pmatrix} \langle y, \mu_T, a \rangle \in \mathcal{R} \text{ and} \\ \langle x, \mu(\cdot \mid T = y), (y, a) \rangle \in \mathcal{R} \end{pmatrix}.$$

# Work in progress

- These axioms are a work in progress.
- However, I can already do cool things with them.
- I have a new randomness reducibility as well.
- It treats randoms as infinitesimally small point masses and compares their relative masses.
- It says, for example, if $x \in 2^{\mathbb{N}}$ is random on the Lebesgue measure, then $0x$ is exactly half as random as $x$.
- There are now more questions than answers.

# Other randomness axioms

- van Lambalgen two related axiomatizations of randomness.
- Alex Simpson is currently developing a set theoretic axiomatization of randomness based on independence.

# Closing Thoughts

# New directions in randomness

- I hope I made you think about algorithmic randomness in new and interesting ways.
- I hope I inspired the poly-time randomness folks and the higher randomness folks to consider how much of this applies to their world.
- I hope those interested in Schnorr and computable randomness found some interesting new theorems.

# Thank You!

These slides will be available on my webpage:

http://www.personal.psu.edu/jmr71/

Or just Google™ me, "Jason Rute".

P.S. I am on the job market.