Multiple recurrence and algorithmic randomness

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Slides are on my web site

Plan

- Algorithmic randomness connects to ergodic theory via an effective study of "almost-everywhere" statements, such as Birkhoff's 1939 theorem:

Let (X, μ, T) be a measure preserving system, and let $f\colon X \to \mathbb{R}$ is measurable. For μ-almost every x, the limit as $N \to \infty$ of the averages of $f \circ T^i(x)$ over $0 \leq i < N$, exists.

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- We address these connections for the multiple recurrence theorem due to Furstenberg (J. Analyse Math., 1977). So far we only do this in the rather special case of shifts on Cantor space.

Some important papers connecting algorithmic randomness with ergodic theory

- V'yugin, TCS, 1999.
  Shows that ML-randomness suffices for the effective Birkhoff theorem. (Note that $T\colon \subseteq X \to X$ only needs to be defined μ-a.e.)
- Franklin and Towsner, Moscow Math. J, recent.
  Sharpness of V'yugin's result.
- Gacs, Hoyrup, Rojas, 2009; Galatolo, Hoyrup, Rojas, 2011.
  General theory of computable probability spaces and computable measure preserving systems; Kolmogorov-Sinai entropy, etc.

Multiple recurrence

Classical theory

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The following is Furstenberg's multiple recurrence theorem (1977); see Furstenberg's book on recurrence, 2014 edition, Thm. 7.15.

Theorem

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $T_1, \ldots, T_k$ be commuting measure preserving operators on X. For each $P \in \mathcal{B}$ with $\mu P > 0$, there is $n > 0$ such that $\mu(\bigcap_i T_i^{-n}(P)) > 0$.

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- In fact, he proves $0 < \liminf_N \frac{1}{N} \sum_{n=1}^{N} \mu(\bigcap_i T_i^{-n}(P))$.
- One can also strengthen to: a.e. $z \in P \; \exists n \, [z \in \bigcap_i T_i^{-n}(P)]$.

Kurtz $\Rightarrow$ k-recurrence in clopen $\mathcal{P}$

In the following we work with $X = \{0,1\}^{\mathbb{N}}$, and the shift operator $S : X \to X$ that takes the first bit off a sequence.

Definition

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be measurable, and $Z \in \{0,1\}^{\mathbb{N}}$. We say that Z is k-recurrent in $\mathcal{P}$ if $S^n(Z), S^{2n}(Z), \ldots, S^{kn}(Z) \in \mathcal{P}$ for some $n \geq 1$, i.e.

$$Z \in \bigcap_{1 \leq i \leq k} S^{-ni}(P).$$

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Proposition

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be clopen, $\mathcal{P} \neq \emptyset$.
Each Kurtz random Z is k-recurrent in $\mathcal{P}$, for each $k \geq 1$.

Proposition (again)

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be clopen, $\mathcal{P} \neq \emptyset$. Let Z be Kurtz random and $k \geq 1$. There is $n \geq 1$ such that $Z \in \bigcap_{1 \leq i \leq k} S^{-ni}(P)$.

Proposition (again)

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ *be clopen,* $\mathcal{P} \neq \emptyset$*. Let* Z *be Kurtz random and* $k \geq 1$*. There is* $n \geq 1$ *such that* $Z \in \bigcap_{1 \leq i \leq k} S^{-ni}(P)$*.*

Suppose there is no such n. We define a null Π^0_1 class $\mathcal{Q}$ containing Z.

- Let n_0 be least such that $\mathcal{P} = [F]^{\prec}$ for some set F of strings of length n_0.
- Let $n_t = n_0(k+1)^t$ for $t \geq 1$.
- Let $\mathcal{Q} = \{Y : \forall t\, [Y \notin \bigcap_{1 \leq i \leq k} S^{-in_t}(\mathcal{P})]\}$. Then $Z \in \mathcal{Q}$.

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By definition of n_0, the classes in the same intersection are independent, so we have for each t

$$\lambda(\{0,1\}^{\mathbb{N}} - \bigcap_{1 \leq i \leq k} S^{-in_t}(\mathcal{P})) = 1 - (\lambda\mathcal{P})^k < 1.$$

The Π^0_1 class $\mathcal{Q}$ is the intersection of independent such classes ranging over all t. Therefore $\mathcal{Q}$ is null.

Schnorr $\Rightarrow$ k-recurrence in Π^0_1 classes with positive computable measure

Theorem

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class with $0 < p = \lambda\mathcal{P}$ a computable real. Each Schnorr random Z is k-recurrent in $\mathcal{P}$, for each $k \geq 1$.

Schnorr ⇒ k-recurrence in Π^0_1 classes with positive computable measure

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This extends the previous argument. For each v we have an error set $G_v \subseteq \{0,1\}^{\mathbb{N}}$. We make the sequence $\langle n_t \rangle$ grow much faster than before:

Let $n_0 = 1$. Let $n = n_t \geq (k+1)n_{t-1}$ be so large that

$$\lambda(\mathcal{P}_n - \mathcal{P}) \leq 2^{-t-v-k}.$$

Define G_v so that $\langle G_v \rangle_{v \in \mathbb{N}}$ is a Schnorr test. If $Z \notin G_v$ for some v, we can apply the independence argument used for Kurtz randomness.

ML randomness $\Rightarrow$ k-recurrence in Π^0_1 classes

Theorem

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class with $0 < \lambda\mathcal{P}$.
Each Martin-Löf random Z is k-recurrent in $\mathcal{P}$, for each $k \geq 1$.

ML randomness $\Rightarrow$ k-recurrence in Π^0_1 classes

Theorem

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class with $0 < \lambda\mathcal{P}$.
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Fix k. First we prove the assertion under the additional assumption that $\mathcal{P}$ is large: $1 - 1/k < \lambda\mathcal{P}$.

For a string η and $u \leq |\eta|$, we write $S^u(\eta)$ for the string η with the first u bits removed.

Each Martin-Löf random Z is k-recurrent in $\mathcal{P}$

Let $B \subseteq 2^{<\omega}$ be a prefix-free c.e. set such that $[B]^{\prec} = \{0,1\}^{\mathbb{N}} - \mathcal{P}$. We define a uniformly c.e. sequence $\langle C^r \rangle$ of prefix free sets.
Let $C^0 = \{\emptyset\}$.
Suppose $r > 0$ and σ is enumerated into C^{r-1} at stage s (so $|\sigma| = s$).

Stage $t > (k+1)s$*:* look for $\eta \succ \sigma$ a minimal string of length t such that $S^{si}(\eta) \in B_t$ for some $i \leq k$. Put η into C^r at stage t.

Let $q = k\lambda[B]^{\prec}$. Then $q < 1$ by hypothesis.
The local measure above σ of the η's we put into C^r is at most q.
Inductively this implies:

For each $r \geq 0$ we have $\lambda[C^r]^{\prec} \leq q^r$.

If Z is not k-recurrent for $\mathcal{P}$ then $Z \in \bigcap_r [C^r]^{\prec}$, so not ML-random.

General case

Theorem (again)

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be a Π^0_1 class with $0 < p = \lambda\mathcal{P}$.
Each Martin-Löf random Z is k-recurrent in $\mathcal{P}$, for each $k \geq 1$.

- If $1 - 1/k \geq \lambda\mathcal{P}$ (i.e., $1/k \leq \lambda[B]^{\prec}$ where $[B]^{\prec}$ is the complement of $\mathcal{P}$), then $\lambda[C^r]^{\prec}$ could easily be 1.
- To remedy this, we choose a finite set $D \subseteq B$ such that the set $\widetilde{B} = B - D$ satisfies $1/k > \lambda[\widetilde{B}]^{\prec}$.
- We modify the argument for the Kurtz case, using the complement of $[D]^{\prec}$ as the clopen set (called $\mathcal{P}$ there).
- If Z passes a ML-test corresponding to the Kurtz test before, then the previous argument works with $\widetilde{B}$.

Recurrence for k shift operators

- The space is $\mathcal{X} = \{0,1\}^{\mathbb{N}^k}$
- For $1 \le i \le k$, the operator $T_i \colon \mathcal{X} \to \mathcal{X}$ takes a row of bits off in direction i.
- Z is recurrent in a class $\mathcal{P} \subseteq \mathcal{X}$ if $\exists n \forall i \, [Z \in T_i^{-n}(P))]$.

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We can modify the methods above to show:

Theorem

Let $\mathcal{P} \subseteq \mathcal{X}$ be a Π^0_1 class with $0 < \lambda\mathcal{P}$. Let $Z \in \mathcal{X}$.
(a) Kurtz (b) Schnorr (c) ML-randomness of Z

implies that Z is recurrent in $\mathcal{P}$ if

(a) $\mathcal{P}$ is clopen (b) $\lambda\mathcal{P}$ is computable (c) for any $\mathcal{P}$.

Recurrence for compact systems

An example of a computable compact system is rotation of S^d of the unit circle C by $2\pi d$, for a computable irrational d. (This is ergodic and not weakly mixing.)

Question

Does multiple recurrence for powers of S_d hold for ML-random $z \in C$ and Π^0_1 classes of positive measure?

This would mean: for each Π^0_1 class $\mathcal{P} \subseteq C$ of positive measure,
for each k,
for a.e. z there is n such that $S_{din}(z) \in \mathcal{P}$ for $1 \leq i \leq k$.

General Conjecture

It is likely that an effective multiple recurrence theorem holds in full generality. If the system is not ergodic, as in the classic case, we have to require that $z \in \mathcal{P}$.

Conjecture

Let (X, μ) be a computable probability space. Let $T_1, \ldots, T_n$ be computable measure preserving transformation that commute pairwise.
Let $\mathcal{P}$ be a Π^0_1 class with $\mu P > 0$.
If $z \in \mathcal{P}$ is ML-random then $\exists n \bigwedge_i T_i^n z \in \mathcal{P}$.
If $\mu\mathcal{P}$ is computable, then Schnorr randomness of z is sufficient.

By the classic result this holds for weakly 2-random z.
To get there for ML-random z, climb the Furstenberg-Zimmer tower?
A draft of this work is available on the 2015 Logic Blog.