Demuth's path to randomness

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joint work with

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Remark

- It is a survey of Demuth's work in the area of
  - constructive analysis
  - algorithmic randomness and computability
- A special focus: differentiability of Markov computable functions, various randomness notions based on various levels of null classes motivated by constructive analysis, results on tt-reducibility and semigenericity
- Some recent developments in Demuth's program: a link between computable analysis and algorithmic randomness

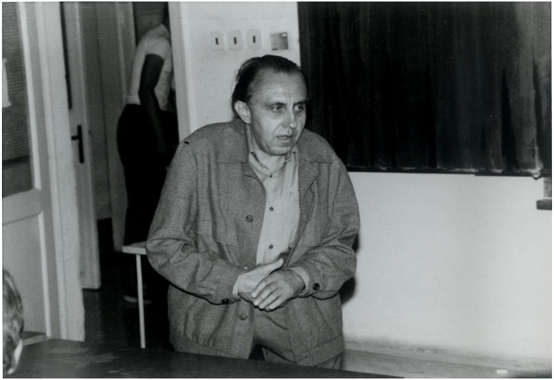


Figure: Osvald Demuth

Osvald Demuth (1936-1988)

Primary interest: constructive analysis in the Russian style

- Master degree: Charles University, Prague, 1959
- (equivalent of) PhD: 1964, Moscow, superviser A.A. Markov
- Habilitation: 1968, Charles University, Prague
- period after 1969:
  - a revenge for his opposition to Russian invasion in 1968
    - a ban to lecture at the university (1972-1978)
    - a ban to travel abroad (1969-1987)
  - nevertheless he was allowed to continue his scientific work at the university

Demuth's constructivism changed over time:
from a strong constructivism working with constructive objects (like computable reals) to larger classes of non-constructive objects (like Δ^0_2 reals, arithmetical reals or even all reals)

Demuth's approach: extended constructivism

RUSS = Russian school of constructive mathematics:

- constructive objects coded by words in a finite alphabet
- constructive interpretation of mathematical propositions ($\exists$, $\vee$)
  double negation elimination permissible only as Markov's principle (for computable functions $\neg\neg f(x)\downarrow \implies f(x)\downarrow$)

Demuth's papers:
written in notationally heavy formal constructive language
We phrase notions and results in the modern language

Computable reals:
computable sequences of rationals with a computable cauchy property

Remark.
Subtle difference:
- *constructively*: a computable real is a finite syntactic object (an index of the sequence)
- *modern approach*: computable reals are those reals having a computable name

$\mathbb{R}_c$ = the collection of computable reals

Markov computable functions:

$g : \mathbb{R}_c \to \mathbb{R}_c$ is Markov computable if from any index of a computable real x one can compute an index of computable real $g(x)$

standard computable functions

$f : \mathbb{R} \to \mathbb{R}$ is computable if

(i) for every computable sequence of reals $(x_k)_{k \in \mathbb{N}}$, the sequence $(f(x_k))_{k \in \mathbb{N}}$ is computable

(ii) f is effectively uniformly continuous

Restriction in what follows. Only Markov computable functions defined on all $\mathbb{R}_c$ and constant outside $[0, 1]$.

Remark

- each Markov computable function is continuous on $\mathbb{R}_c$ (Ceĭtin or Kreisel, Lacombe Shoenfield)
- Markov computable functions are not necessarily uniformly continuous

  (hint: take a Σ^0_1 class S containing all computable reals with the complement of S nonempty and define f piecewise linear on intervals from S with growing maximum at midpoints of these intervals. Thus, f takes arbitrarily large values at computable reals close to reals $r \notin S$).

Notation.
$\emptyset^{(n)}$-uniform continuity means uniform continuity where a modulus of uniform continuity can be computed by $\emptyset^{(n)}$

Remark

- $\emptyset$-uniformly continous MC-functions can be obtained by restricting standard computable functions from $\mathbb{R}$ to $\mathbb{R}_c$
- standard computable functions can be obtained by extending $\emptyset$-uniformly continuous MC-functions from $\mathbb{R}_c$ to $\mathbb{R}$

Notions of randomness in Demuth's work.

Demuth did not use notion "randomness". He used

"non-approximibility in measure" instead of "randomness"
"approximibility in measure" instead of "non-randomness"

randomness:
in the context of probability, statistics, information theory

non-approximibility in measure:
in the context of constructive analysis

Survey of randomness notions studied by Demuth

- B-measure zero (and B-measurability)
  equivalent to B-Schnorr tests (and relevant measurability)
- Denjoy randomness
  equivalent to Computable Randomness, (for Denjoy alternative for $\emptyset$-uniformly continuous Markov computable functions)
- non-approximibility in measure, NAP-sets
  equivalent to ML-randomness, (for differentiability of Markov computable functions of bounded variation)
- non-weakly approximibility in measure, NWAP-sets
  now Demuth randomness (for Denjoy alternative for Markov computable functions)
- NWAP*-sets
  now weak Demuth randomness (for computability theory)

Demuth 1969: a notion "for almost every computable real"
He used sequences of (so called) S_σ sets.
It is equivalent to Schnorr tests.

Later (extended constructivism) he defined relativized version of measure zero sets of reals and also the notion of relativized measure of sets of reals. The concept of B-measure zero set of reals is defined by means of B-Schnorr tests (i.e. Schnorr-tests relativized to B).

Computable reals do not suffice to study the points of differentiability of Markov computable functions.
More reals are needed !

(There is an absolutely continuous Markov computable function not pseudo-differentiable at any computable real.)

Demuth 1975: introduced a notion equivalent to ML-randomness (not aware of 1966 paper by Martin-Löf). First for Δ^0_2 reals, later for arithmetical reals and eventually for all reals.

Definition (Demuth 1975)

- A Δ^0_2 real x is a Π_1-*number* if there is a computable sequence of rationals $(q_n)_{n\in\mathbb{N}}$ with $x = \lim_{n\to\infty} q_n$ and a computable sequence of finite computable sets $(C_m)_{m\in\mathbb{N}}$ such that $\lambda(\bigcup_{n\notin C_m}[q_n, q_{n+1}]) < 2^{-m}$.
- Π_2-numbers are those Δ^0_2 reals which are not Π_1-numbers.

Facts.

- Π_2-numbers are exactly Δ^0_2 reals which are ML-random.
- Π_2-numbers are exactly Δ^0_2 reals which are finitely bounded random reals as defined by Brodhead, Downey, Ng.

Some of Demuth's results (1975)

- There is a universal ML-test (in a different terminology)
- Using this test he later extended "ML-randomness" from Δ^0_2 reals to first $\mathcal{A}_2$-numbers (arithmetical ML-random reals) and yet later to all ML-random reals.
- Equivalence of Solovay randomness and ML-randomness (independently: a notion of Solovay tests).

Algebraic operations on Π_1-numbers and Π_2-numbers

Theorem (Demuth 1975)

- *Any Δ^0_2 real can be expressed as the sum of two Δ^0_2 non-ML-random reals*

  but

- *If α is left-.c.e. and ML-random, and $\alpha = \beta_1 + \beta_2$ for left-c.e. reals β_1, β_2 then at least one of them (β_1 or β_2) is also ML-random.*

Theory of left-c.e. reals (developed by Solovay): an area of much interest in recent years.

Differentiability of Markov computable functions.

Problem: such functions are only defined on the computable reals.
Solution: upper and lower "pseudo-derivatives" at a real z, taking the limit of slopes close to z where the function is defined.

Let g be a function defined on $I_{\mathbb{Q}}$ (the rationals in $[0,1]$), $z \in [0,1]$. Let

$$\widetilde{D}g(z) = \limsup_{h \to 0^+} \{S_g(a,b) \colon a, b \in I_{\mathbb{Q}} \wedge \ a \le z \le b \wedge \ 0 < b - a \le h\}.$$

$$\underset{\sim}{D}g(z) = \liminf_{h \to 0^+} \{S_g(a,b) \colon a, b \in I_{\mathbb{Q}} \wedge \ a \le z \le b \wedge \ 0 < b - a \le h\}.$$

g is pseudo-differentiable at x if $\quad -\infty < \underset{\sim}{D}g(x) = \widetilde{D}g(x) < \infty$, in which case the value $\underset{\sim}{D}f(x) = \widetilde{D}f(x)$ will be denoted $f'(x)$.

Theorem (Demuth 75)

Let f be a Markov computable function of bounded variation.

(i) *f is pseudo-differentiable at any Δ^0_2 Martin-Löf random real.*

(ii) *Furthermore, there is a Schnorr test relative to $\emptyset'$ such that for any Δ^0_2 real ξ passing the test, ξ is Martin-Löf random, $f'(\xi)$ exists, and $f'(\xi)$ is a Δ^0_2 real which can be computed uniformly in $\emptyset'$ and the representation of ξ as a Δ^0_2 real.*

Warning: a Markov computable function of bounded variation

- need not be expressible as a difference of two non-decreasing Markov computable functions (Ceĭtin and Zaslavskiĭ)
- is expressible as $f_1 - f_2$, where f_1, f_2 are non-decreasing *interval-c.e. functions*, but unfortunately, functions of this more general type need not be differentiable at each ML-random real (proved by Nies)
- a different and more complicated approach is needed.

Idea:

(i) f is first truncated into a sequence of Lipschitz functions. By approximations of Lipschitz functions by polygonal functions it is proved that any Markov computable Lipschitz function is pseudo-differentiable at any ML-random real.

Then it is shown that outside of ML-null class of reals it yields pseudo-differentiability of f.

As to (ii), it uses the fact that there is a single Schnorr test (even Demuth test - see later) such that any z passing the test is GL_1.

Remark:

Brattka, Miller, Nies proved a generalization of (i) for all ML-random reals (not only Δ^0_2).

Definition (Demuth 1978, 1980)

We say that the *Denjoy alternative* holds for a (partial) function f at $z \in [0,1]$ if

$$\text{either } \widetilde{D}f(z) = \underset{\sim}{D}f(z) < \infty,$$

$$\text{or } \widetilde{D}f(z) = \infty \text{ and } \underset{\sim}{D}f(z) = -\infty.$$

Definition (Demuth 1978, 1980)

A real $z \in [0,1]$ is called *Denjoy random* (or a *Denjoy set*) if for no Markov computable function g do we have $\underset{\sim}{D}g(z) = \infty$.

Well-known fact from classical analysis (Denjoy, Young, Sacks):

For any function $g\colon [0,1] \to \mathbb{R}$, the reals z such that

$$\underline{D}g(z) = \infty \text{ form a null set.}$$

(Cater: a stronger fact. The reals z where the right lower derivative $\underline{D}_+g(z)$ is infinite form a null set.)

Theorem (Demuth 1978)

If $z \in [0,1]$ is Denjoy random, then for every $\emptyset$-uniformly continuous Markov computable function $f\colon [0,1] \to \mathbb{R}_c$ the Denjoy alternative holds at z.

Combining this with the results of Bienvenu, Brattka, Hölzl, Miller, Nies it yields a characterization of computable randomness through differentiability of standard computable functions.

Theorem (Bienvenu, Hölzl, Miller, Nies, 2014)

The following are equivalent for a real $z \in [0,1]$.

(i) *z is Denjoy random.*

(ii) *z is computably random*

(iii) *for every standard computable $f\colon [0,1] \to \mathbb{R}$ the Denjoy alternative holds at z.*

Remark

Demuth studied computable randomness indirectly via Denjoy randomness (Denjoy sets).

Example of his result.

Theorem (Demuth 1988)

Every Denjoy random set that is non-ML-random must be high.

Nies, Stephan, Terwijn (2005) (independently) rediscovered this result and proved a kind of converse:

Each high degree contains a computably random set which is not ML-random.

Question:

How much randomness is needed for Denjoy alternative for all Markov computable functions? ML-randomness is not sufficient.

Theorem (Demuth 1976)

There is a Markov computable function f such that the Denjoy alternative fails at some ML-random real z. Moreover, f is extendable to a continuous function on $[0, 1]$.

This theorem has been reproved by Bienvenu, Hölzl, Miller and Nies and even strengfthed: one can make such a z left-c.e.

Stronger randomness is needed ("Demuth randomness").

Definition (Demuth 1982)

- A Demuth test is a sequence of c.e. open sets $(S_m)_{m\in\mathbb{N}}$ such that $\forall m\, \lambda(S_m) \le 2^{-m}$, and there is a function $f\colon \mathbb{N} \to \mathbb{N}$ with $f \le_{\mathrm{wtt}} \emptyset'$ such that $S_m = [W_{f(m)}]^{\prec}$.
  A set Z *passes* the test if $Z \notin S_m$ for almost every m. We say that Z is Demuth random if Z passes each Demuth test.

- We say that a set $Z \subseteq \mathbb{N}$ is weakly Demuth random if for each Demuth test $(S_m)_{m\in\mathbb{N}}$ there is an m such that $Z \notin S_m$.

Remark

A set Z is weakly Demuth random if and only if Z passes all monotonic Demuth tests, i.e. Demuth tests $(S_m)_{m\in\mathbb{N}}$ such that $S_m \supseteq S_{m+1}$, for each m.

а) $S_0(q) \rightleftharpoons \forall k\ell\,(!\langle q\rangle(k,\ell)\ \&\ !\lim\limits_{t\to\infty}\langle q\rangle(k,t))$,

$\hat{S}_0(q) \rightleftharpoons (S_0(q)\ \&\ \forall k\,(\hat{\mu}_1(\mathcal{L}im\,(s_1^1(q,k))) \le 2^{-k}))$,

$\bar{S}_0(q) \rightleftharpoons (S_0(q)\ \&\ \forall k\,(\mu_0(\mathcal{L}im\,(s_1^1(q,k+1))) \le 2^{-k-1}))$,

$\mathcal{K}(p,q) \rightleftharpoons (\mathcal{K}_0(q)\ \&\ \forall k\,(!\langle p\rangle(k)\ \&\ \mathcal{M}is\,(s_1^1(q,k)) \le \langle p\rangle(k)))$,

где $\mathcal{K}$ одно из выражений S , $\hat{S}$ и $\bar{S}$,

б) если верно $S_0(q)$, то

$$\mathcal{V}_q \rightleftharpoons \bigcap_m \widehat{\bigcup_{n\ge m}} [W_{\mathcal{L}im\,(s_1^1(q,n))}],$$

$$\mathcal{V}_q^* \rightleftharpoons \widehat{\bigcup_m} \bigcap_{n\ge m} [W_{\mathcal{L}im\,(s_1^1(q,n))}],$$

$$\mathcal{T}_q \rightleftharpoons \widehat{\bigcup_k} [\mathcal{D}_{\mathcal{L}im\,(s_1^1(q,k))}]_c,$$

в) ${}^p\mathcal{A}_\alpha \rightleftharpoons \wedge X\,(\neg\neg\,\exists m\,(\hat{S}(p,m)\ \&\ X\in\mathcal{V}_m))$;

г) $\mathcal{A}_\alpha \rightleftharpoons \wedge X\,(\neg\neg\,\exists pq\,(\hat{S}(p,q)\ \&\ X\in\mathcal{V}_q))$,

$\mathcal{A}_\alpha^* \rightleftharpoons \wedge X\,(\neg\neg\,\exists p\,q\,(\hat{S}(p,q)\ \&\ X\in\mathcal{V}_q^*))$,

$\mathcal{A}_\beta \rightleftharpoons \mathcal{A} \setminus \mathcal{A}_\alpha$.

Figure: $\mathcal{A}_\beta$ is the definition of Demuth randomness (1982)

Theorem (Demuth 1983)

Let z be a Demuth random real. Then the Denjoy alternative holds at z for every Markov computable function.

Remark (Bienvenu, Hölzl, Miller and Nies, 2012)

i) difference randomness (introduced by Franklin and Ng) is sufficient as a hypothesis on the real z in the above Theorem

ii) the "randomness notion" to make the Denjoy alternative hold for each Markov computable function is incomparable with ML-randomness!

Computability-theoretic properties of Demuth randomness

Theorem (Demuth 1982, 1988)

For any Demuth random z

- $z' \equiv_{\emptyset'-tt} z$
  and, thus,
- $z' \equiv_T z \oplus \emptyset'$ *(i.e.* $z \in GL_1$*)*

Warning: $z' \equiv_{\emptyset'-tt} z$ does not imply $z' \equiv_{tt} z \oplus \emptyset'$
(due to a computable bound on the use of $\emptyset'$ in a tt-reduction).

Theorem (Demuth 1982, 1988)

There is a single Demuth test $(S_m)_{m \in \mathbb{N}}$ *such that for every z for which* $z \in S_m$ *for at most finitely many m, z is generalized low.*

Theorem (Demuth 1988)

Let y be Demuth random and x ML-random. If $x \leq_T y$ then x is Demuth random.

A variant of this Theorem was only recently rediscovered.

Miller, Yu (2008): for every 2-random y and ML-random x, $x \leq_T y$ implies that x is 2-random.
More generally: it holds for any z-ML-random instead of 2-random.

Idea of these proofs: For a Turing functional Φ and $n > 0$ let

$$S^A_{\Phi,n} = [\{\sigma \in \{0,1\}^* \colon\ A \upharpoonright n \preceq \Phi^\sigma\}]^\prec.$$

Demuth and K. (1987), and independently Miller and Yu (2008) proved that if A is Martin-Löf random then there is a constant c such that $\forall n\, \lambda(S^A_{\Phi,n}) \leq 2^{-n+c}$.

Theorem (Demuth 1988)

For any $z \geq_T \emptyset'$, there is a Demuth random x such that $x' \equiv_T z$.

Corollary

There exists a Δ^0_2 Demuth random real.

The above theorem is proved by using the following strong result.

Theorem (Demuth 1988)

For any y, z, any $\mathcal{E} \subseteq [0,1]$ of y-measure zero, there is a $x \notin \mathcal{E}$ such that $x \leq_T y \oplus z$ and $z \leq_T x \oplus y$.

Theorem (Demuth 1988)

There is a $\emptyset'$-computable function g such that for every Demuth random z and every z-partial computable function f, $f(n) \leq g(n)$ for almost every n.

Kurtz 1981: $\emptyset'$ is uniformly almost everywhere dominating, (i.e. there is a $\emptyset'$-computable function dominating every total function computable from a member of a measure one set of reals $\mathcal{S}$). Demuth (unaware of this result) improved it in two ways,

(1) by showing that $\mathcal{S}$ includes every Demuth random real, and
(2) by showing the function g dominates every partial function computable from every Demuth random.

Kjos-Hanssen, Miller, Solomon, 2012:
C is UAED if and only if $C \geq_{LR} \emptyset'$.

Demuth randomness and lowness notions

Examples of recent results:

(i) K, Nies, 2011: every c.e. set Turing below a Demuth random is strongly jump traceable
Greenberg, Turetsky, 2014: every c.e. strongly jump traceable set has a Demuth random set Turing above

(ii) Nies, 2012: each base for Demuth randomness is strongly jump traceable.
Greenberg, Turetsky, 2014: This inclusion is proper.

Recall that the lower density of a measurable set $\mathcal{P}$ at a real z is

$$\rho(\mathcal{P}|z) = \liminf_{h \to 0} \{\lambda(\mathcal{P} \cap I)/\lambda(I) : I \text{ is an open interval, } z \in I \ \& \ |I| < h\}.$$

Definition

A real z is

- a density-one point if for every effectively closed class $\mathcal{P}$ containing z, $\rho(\mathcal{P}|z) = 1$.
- a positive density point if for every effectively closed class $\mathcal{P}$ containing z, $\rho(\mathcal{P}|z) > 0$.

Theorem (Demuth 1982)

Every weakly Demuth random is a density-one point.
Moreover, there is a single Demuth test $(S_m)_{m \in \mathbb{N}}$ such that every real for which $z \notin S_m$ for infinitely many m is a density-one point.

A slightly stronger randomness than Demuth randomness

Definition (Bienvenu, Greenberg, K., Nies, Turetsky, s:2013)

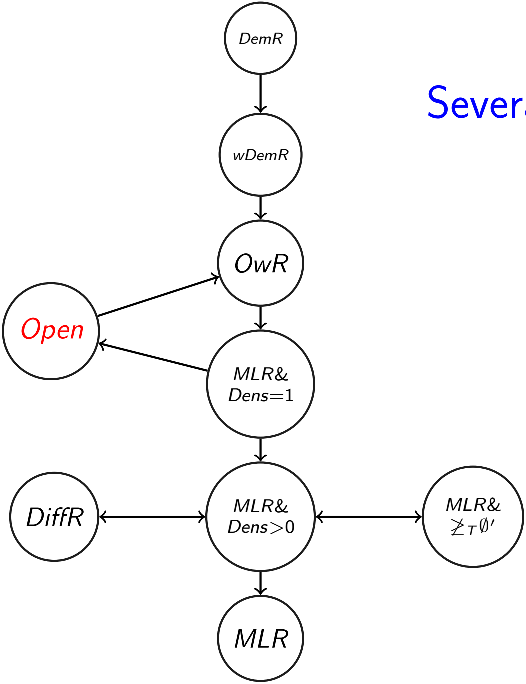
(i) A left-c.e. bounded test is an effective descending sequence $(U_m)_{m\in\mathbb{N}}$ of effectively open sets in $[0,1]$ together with computable increasing sequence of rationals $(\beta_m)_{m\in\mathbb{N}}$ with limit β such that $\lambda(U_m) \leq \beta - \beta_m$ for every m.

(ii) A real z is Oberwolfach random if and only if it passes every left-c.e. bounded test.

Theorem (Bienvenu, Greenberg, K., Nies, Turetsky, s:2013)

Every Oberwolfach random is a density-one point.

Several authors



Importance of density-one points and positive density points

Notions of density-one and positive density were used for a solution of so-called "covering problem"

Theorem (Group 1 and Group 2, 2014)

A set A is K-trivial if and only if there is a set Z which is ML-random, $A <_T Z$ *and* $Z \not\geq_T \emptyset'$.

Group 1: Bienvenu, Greenberg, K, Nies, Turetsky
(Every K-trivial is Turing below any ML-random which is not Oberwolfach random)
Group 2: Day, Miller
(There is ML-random which is of positive density but not of density 1).

Reducibilities based on constructive analysis

By R we denote an operator mapping a Markov computable function g to the maximal continuous extension $R[g]$ of g.

Definition (Demuth 1988)

Given $\alpha, \beta \in [0, 1]$,

- α is f-reducible to β, denoted $\alpha \leq_f \beta$, if there is a Markov computable function g such that $R[g](\beta) = \alpha$.
- α is $\emptyset\text{-ucf}$-reducible to β, denoted $\alpha \leq_{\emptyset\text{-ucf}} \beta$, if α is f-reducible to β via a Markov computable function g that is $\emptyset$-uniformly continuous.
- α is mf-reducible to β, denoted $\alpha \leq_{\mathrm{mf}} \beta$, if α is f-reducible to β via a Markov computable function g that is monotonically increasing.

To compare these reducibilities to those from computability theory one needs to exclude a set $\mathcal{C} \subseteq 2^{\mathbb{N}}$ of $\emptyset$-measure zero that contains all finite and cofinite sequences.

Theorem (Demuth 1988)

1. *For any $\emptyset$-uniformly continuous Markov computable function f, one can uniformly obtain an index of a tt-functional Φ such that for every $A, B \in 2^{\mathbb{N}}$ such that $B \notin \mathcal{C}$,*

$$A \leq_{\emptyset\text{-ucf}} B \text{ via } f \text{ if and only if } A \leq_{\mathrm{tt}} B \text{ via } \Phi.$$

2. *For any tt-functional Φ, one can uniformly obtain the index of a $\emptyset$-uniformly continuous Markov computable function $f : [0,1] \to [0,1]$ such that for any $A, B \in 2^{\mathbb{N}}$ such that $A, B \notin \mathcal{C}$*

$$A \leq_{\emptyset\text{-ucf}} B \text{ via } f \text{ if and only if } A \leq_{\mathrm{tt}} B \text{ via } \Phi.$$

The following theorem is known as "Demuth's theorem".

Theorem (Demuth 1988, "Demuth's theorem")

If B is non-computable and tt-reducible to a ML-random A, then there is a ML-random C such that

$$B \leq_{\mathrm{tt}} C \leq_T B.$$

Demuth's result is, in a sense, the best possible.

Theorem (Bienvenu, Porter, 2012)

There is a ML-random A and a tt-functional Φ such that $\Phi(A)$ is non-computable and cannot wtt-compute any ML-random.

Recent proofs (following Kautz): in terms of computable measures.

Demuth's proof takes a different approach based on $\emptyset$-ucf-reduction and mf-reduction:

1) tt-reduction Φ is replaced by an $\emptyset$-ucf reduction by by some Markov computable function f
2) From this function f a set C is constructed such that $B \leq_{\mathrm{mf}} C$ by a monotone Markov computable function g
3) mf-reduction by g yields tt-reduction from B to C.

Close examination shows: the function g witnessing the mf-reduction is the distribution function of the computable measure induced by the initial tt-functional Φ.
Thus, both proofs are, in fact, similar.

Definition (Demuth 1987)

A non-computable set Z is called semigeneric if every Π^0_1 class containing Z has a computable member (i.e. every Σ^0_1 class containing every computable set contains also Z).

Semigenericity means to be close to computable (the set cannot be separated from the computable sets by a Π^0_1 class).

The notion was later studied by J. Miller (2002) (unavoidability) and by Kalantari and Welch (2003) (shadow points).

Semigenericity plays an important role in constructive analysis.

Remark. For every Markov computable function g, the classical extension $R[g]$ of g is continuous at every semigeneric real.

Definition

- Ceĭtin 1970: A set Z is strongly undecidable if there is a partial computable function p such that for any computable set M and any index v of its characteristic function, $p(v)\downarrow\ \&\ Z \upharpoonright p(v) \neq M \upharpoonright p(v)$.
- Miller 2002: If the function p is total Z is hyperavoidable.

Theorem (Demuth, K. 1987; Miller 2002)

A non-computable set Z is semigeneric if and only if Z is not strongly undecidable.

Remark.

- unavoidable = computable or semigeneric
- avoidable = strongly undecidable =
  = non-computable and non-semigeneric
- hyperavoidable =
  = non-computable and non-weakly semigeneric
  (defined next)

Easy fact.

A non-computable set A is semigeneric if A belongs to any Σ_1^0 class $[W_a]^{\prec}$ for which there is a partial computable function p such that for any computable set M and any index v of M, $p(v)\downarrow$ & $\sigma \preceq M \upharpoonright p(v)$ for some string $\sigma \in W_a$.

Definition

A non-computable set A is weakly semigeneric if A belongs to any Σ_1^0 class $[W_a]^{\prec}$ for which there is a total computable function h such that for any e if $\varphi_e(j)\downarrow$ and $\varphi_e(j) \leq 1$ for all $j < h(e)$ then for $\tau = \varphi_e(0) \ldots \varphi_e(h(e) - 1)$ we have $\sigma \preceq \tau$ for some string $\sigma \in W_a$.

Theorem (Kjos-Hanssen, Merkle, Stephan)

Hyperavoidable sets are equivalent to complex sets

i.e. sets X, $C(X \upharpoonright n) \geq f(n)$ for some computable order f.

Theorem (Demuth 1987)

1. *No semigeneric real can* tt*-compute a ML-random real.* *(Bienvenu, Porter: the converse does not hold)*
2. *If a set* Z *is semigeneric then any set* B *such that* $\emptyset <_{\mathrm{tt}} B \leq_{\mathrm{tt}} Z$ *is also semigeneric (non-computable semigeneric reals are closed downwards under* tt*-reducibility).*

Item (2): similarity with Demuth's theorem. The class of non-computable reals that are ML-random with respect to some computable measure is closed downwards under tt-reducibility.

A similar result to (1):

Theorem (Demuth, K., 1987)

No 1-generic can T*-compute a ML-random.*

Demuth also established some connections between semigenericity and Denjoy randoms (computable randoms):

Theorem (Demuth 1990)

- *every non-ML-random Denjoy random real is high*
- *every real of high degree can compute a semigeneric Denjoy random real*
- *there is a minimal Turing degree containing a semigeneric Denjoy random real*
- *every semigeneric Denjoy random real is* tt*-reducible to a Denjoy random that is neither semigeneric nor ML-random.*

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Thank you