Differentiability of computable monotone functions in $\mathbb{R}^n$

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Known results on the unit interval

Theorem (BMN '15, FKN '14)

$z \in [0,1]$ is computably random
$\iff$ *every computable Lipschitz $f : [0,1] \to \mathbb{R}$ is differentiable at z*
$\iff$ *every computable monotone $f : [0,1] \to \mathbb{R}$ is differentiable at z.*

Main idea was to use the following correspondences:

1. computable martingales $\leftrightarrow$ computable measures

$$\mu(\sigma) = 2^{-|\sigma|}M(\sigma)$$

2. computable measures $\leftrightarrow$ computable monotone functions on the unit interval

$$f(x) = \mu([0;x])$$

Lipschitz functions on $\mathbb{R}^n$

Theorem (Rademacher, 1919)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function. There exists a null set, such that f is differentiable outside it.

The converse question

Let $N \subset \mathbb{R}^n$ be a null set. Is there a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^m$ that is not differentiable inside N?

- positive for $m = n = 1$;
- for $n \geq 2$ there is a null set N such that every Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at some point in N (Preiss 1980);
- A comprehensive answer has been given/announced very recently, not all relevant results have been published yet (Alberti, Csörnyei, Preiss, Speight, Jones, etc);
- In general, the converse holds if and only if $m \geq n$.

Monotone functions on $\mathbb{R}^n$

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a function. We say f is *monotone* if

$$\langle f(x) - f(y), x - y \rangle \geq 0 \text{ for all } x, y \in \mathbb{R}^n.$$

Theorem (Mignot, 1976)

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is monotone, then it is almost everywhere differentiable.

The converse question has not been studied.

Main result

What we have:

Theorem (Galicki & Turetsky, 2014)

$z \in [0,1]^n$ is computably random
$\implies$ *every computable Lipschitz $f : [0,1]^n \to \mathbb{R}$ is differentiable at z.*

What we will show:

Theorem (GT 2014, G 2015)

$z \in \mathbb{R}^n$ is computably random
$\iff$ *every computable monotone $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at z.*

Minty Parameterization

Minty showed that the so called Cayley transformation

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \text{ defined by } \Phi(x, y) = \frac{1}{\sqrt{2}}(y + x, y - x)$$

transforms the graph of a monotone function into a graph of a 1-Lipschitz function.

Proposition (A)

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be monotone. Then $(u + I)$ and $(u + I)^{-1}$ are monotone and $(u + I)^{-1}$ is 1-Lipschitz.
Let $z \in \mathbb{R}^n$ and define $f = (u + I)^{-1}$ and $\hat{z} = u(z) + z$. The following two are equivalent:

1. *u is differentiable at z, and*
2. *f is differentiable at $\hat{z}$ and $f'(\hat{z})$ is invertible.*

The “easy” direction ($\Longrightarrow$)

Proposition (A) repeated

Let $z \in \mathbb{R}^n$ and define $f = (u + I)^{-1}$ and $\hat{z} = u(z) + z$. The following two are equivalent:

1. u is differentiable at z, and
2. f is differentiable at $\hat{z}$ and $f'(\hat{z})$ is invertible.

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a monotone computable function and let $z \in [0, 1]^n$ be computably random.

- $g = u + I$ is monotone and computable and
- $f = g^{-1}$ is 1-Lipschitz and computable.
- If we can show that $\hat{z} = g(z)$ is computably random, then f is differentiable at $\hat{z}$.
- By Proposition (A), if $f'(\hat{z})$ is invertible, then g is differentiable at z.

The hard direction ($\Leftarrow$)

Given $z \in \mathbb{R}^n$ not computably random, we need to find a computable monotone function not differentiable at z.

On the real line

- computable martingale M succeeding on Z
- we define computable measure on $[0, 1]$ by $\mu(\sigma) = 2^{-|\sigma|}M(\sigma)$
- let $f(x) = \mathbf{cdf}_\mu(x) = \mu([0; x])$.

To make f both Lipschitz and monotone, the idea was to make M bounded from below and from above while still not converging on Z.

Optimal transport

- we want to transfer resources from one location X to another Y
- we model this by considering probability measures: μ on X and ν on Y
- the cost of transporting $x \in X$ to $y \in Y$ is modelled by some cost function $c : X \times Y \to \mathbb{R}$,
- we are interested in the functions T that push μ onto ν, that is $\nu(A) = \mu(T^{-1}(A))$ for all A. In symbols, $\nu = T\#\mu$.

Monge's optimal transportation problem

Minimize $I[T] = \int_X c(x, T(x))d\mu$ over the set of all measurable $T : X \to Y$ such that $\nu = T\#\mu$.

Reinterpreting the one-dimensional case

Fact (already known to Hoeffding and Fréchet)

Let μ, ν be two probability measures on $\mathbb{R}$, with respective cumulative distribution functions F and G. Assume μ is atomless.
Then $T = G^{-1} \circ F$ is an optimal transport map (that transports μ onto ν) with respect to the quadratic cost $c(x, y) = |x - y|^2$.

- if we consider $\mu(\sigma) = 2^{-|\sigma|}M(\sigma)$, and λ, the optimal transport map T from μ onto λ is given by the cdf of μ.

The idea we'd like to exploit

- z not computably random $\rightarrow$
- a martingale M diverging on Z $\rightarrow$
- a measure μ_M "oscillating" around z $\rightarrow$
- a transfer map T from μ_M onto λ not differentiable at z.

Optimal transport in higher dimensions

Theorem (Knott-Smith '87, Brenier '87-'91, McCann '95)

Let μ, ν be (nice) probability measures on $\mathbb{R}^n$. There exists a unique gradient of a convex function $\nabla\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nabla\phi$ is the optimal transport map from μ onto ν with respect to the quadratic cost.

Theorem (Volume distortion, McCann '97 ??)

Let ϕ be a convex function on $\mathbb{R}^n$ and suppose it is twice differentiable at $x \in \mathbb{R}^n$. Then

$$\lim_{r\to 0} \frac{\lambda\left(\partial\phi(B_r(x))\right)}{\lambda\left(B_r(x)\right)} = \det D_A^2\phi(x).$$

- in our case, the limit on the left is actually $D_\lambda\mu(z)$.
- Consequence: if $D_\lambda\mu(z)$ does not exist, then $\nabla\phi$ is not differentiable at z.

Outline of the proof

- define a computable martingale M diverging on Z
- show that $D_\lambda \mu_M(z)$ does not exists
- show that the optimal monotone transport map from μ_M onto λ is computable

Few points regarding the proof

Characterisation of computable randomness

$z \in [0,1]^n$ is computably random $\iff$ every absolutely continuous computable probability measure on $[0,1]^n$ is differentiable at z

Theorem (Effective Brenier's Theorem)

Let μ, ν be two absolutely continuous computable probability measures on $\mathbb{R}^n$. Then there is a computable convex function ϕ such that $\nabla\phi$ is the optimal monotone transportation map from μ onto ν.

A trouble: this only gives us an almost everywhere computable monotone function!

The last ingredient of the proof for the “hard” direction

- we need $\nabla\phi$ to be computable, not just a.e. computable
- ensuring Hölder continuity of $\nabla\phi$ would do
- Cafarelli’s regularity theory is a series of results of the form: given “nice” properties of μ, ν ensure some continuity properties of ϕ

Theorem (Cafarelli)

Let ϕ be an Aleksandrov solution of

$$\det D^2\phi = h.$$

If h is bounded from above and below by some positive constants, then $\phi \in C^{1,\alpha}$ for some universal exponent α.

In our case the equation is $\det D^2\phi = D_\lambda\mu$. The idea is to make our martingale bounded from below and from above, not converging on Z in such a way as to ensure $D_\lambda\mu(z)$ does not exist.

Monge-Ampére equation

A general form is

$$\det D^2\phi(x) = F(x, \phi(x), \nabla\phi(x))$$

McCann showed that under certain conditions a special case of this equation holds a.e.

Theorem (G 2015)

$z \in \mathbb{R}^n$ is computably random $\iff$ the following equation holds for all computable absolutely continuous probability measures (on $\mathbb{R}^n$) μ

$$\det D^2\phi_\mu(z) = D_\lambda\mu(z).$$

Where ϕ_μ is the optimal transport from μ onto λ.

Effective Aleksandrov Theorem

The result for monotone functions provides also the “converse” direction for the effective Aleksandrov Theorem.

Theorem (G 2015)

$z \in \mathbb{R}^n$ is computably random $\iff$ every computable convex function $\phi : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at z.

Remaining questions

- the converse direction for the Effective Rademacher Theorem
- polynomial setting
- various natural questions concerning computability and theory of optimal transport:
  - computability, algorithmic randomness and Monge-Ampére equation
  - polynomial time version of Brenier's theorem