Bases and *JT*- and *LR*-reducibility

Johanna Franklin

Hofstra University

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Outline

What happens when we frame lowness notions associated with Turing reducibility in the context of other reducibilities?

- Reducibilities
  - LR-reducibility
  - JT-reducibility
- Lowness properties
- JT-bases
- LR-bases

Reducibilities on $\mathcal{P}(\mathbb{N})$

We use reducibilities to classify the elements of $\mathcal{P}(\mathbb{N})$ based on

- whether information given about one real naturally produces information about another ($\leq_T$, $\leq_{tt}$, $\leq_e$, etc.),
- whether one real contains more information than another ($\leq_{LR}$, $\leq_K$, etc.), or
- whether one problem is harder to solve than another (mass problems, complexity theory).

Classical reducibilities: Properties

Classical reducibilities tend to be defined in terms of an underlying map that induces the reduction:

> *$A \leq_T B$ iff there is a computable functional $\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $\Phi(A) = B$.*

These maps are usually effective and simply defined (this one is Σ^0_3).

Weak reducibilities: Properties

A weak reducibility $\leq_W$ should be

- weaker than Turing reducibility:

$$A \leq_T B \Longrightarrow A \leq_W B$$

- simply definable (Σ^0_n as a relation on sets), and
- $X' \not\leq_W X$ for any X.

(Suggested by Nies)

Examples: $\leq_{LK}$, $\leq_{LR}$, $\leq_{JT}$, $\leq_{SJT}$

JT-reducibility

- A B-trace with bound h is a uniformly B-r.e. sequence $\langle V_n^B \rangle_n$ such that for every n,

$$|V_n^B| \leq h(n).$$

- A B-trace $\langle V_n^B \rangle_n$ traces a partial function ψ if for every n,

$$\psi(n) \downarrow \Longrightarrow \psi(n) \in V_n^B.$$

Definition

$A \leq_{JT} B$ if every partial A-recursive function ψ^A is traced by some B-trace with a computable bound h.

Basic facts

- If $A \leq_{JT} \emptyset$, we say A is jump traceable.
- Every jump-traceable set is GL_1.
- An r.e. set is jump traceable iff it is superlow (Nies).
- Jump traceability and superlowness are incomparable notions within the ω-r.e. sets (Nies).
- If $\emptyset' \leq_{JT} A$, then A is JT-hard. If A is Δ^0_2, then this is equivalent to A being superhigh (Simpson).

LR-reducibility

Definition

$A \leq_{LR} B$ if every B-random set is A-random ($\mathsf{ML}^B \subseteq \mathsf{ML}^A$).

Basic facts

- $A \leq_{LR} \emptyset$ iff A is K-trivial.
- There are lots of results (and questions) about the LR-degrees (see Barmpalias).
- If $\emptyset' \leq_{LR} A$, then A is uniformly almost everywhere dominating (Kjos-Hanssen, J. Miller, and Solomon).

LR- and *JT*-reducibilities

Lemma

$A \leq_{LR} B \Longrightarrow A \leq_{JT} B.$

The proof uses a characterization of Kjos-Hanssen, J. Miller, and Solomon and the fact that a proof that lowness for randomness implies jump traceability ($A \leq_{LR} \emptyset \Longrightarrow A \leq_{JT} \emptyset$) relativizes nicely.

Lowness properties

A *lowness property* is a property asserting that a set A resembles $\emptyset$ in some way.

Turing reducibility: A is low iff $A' \equiv_T \emptyset'$.

If $\leq_W$ is based in relativization, then $A \leq_W \emptyset$ means that A is low in the sense of W.

Bases for randomness

If we say A is low if it is easy to compute, we find the following:

Theorem (Sacks)

A is nonrecursive if and only if $\{Z \mid Z \geq_T A\}$ is null.

This means being null is too coarse a measure.

Theorem (Kučera-Gács)

Every set is wtt-reducible to a Martin-Löf random set.

Therefore, an upper cone cannot even be Martin-Löf null.

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Definition (Kučera)

A is a Turing base for randomness if $A \leq_T Z$ for some A-random Z.

Thus, the cone above A is Martin-Löf null relative to A iff A is not a base.

Turing bases for randomness

- The Turing bases for Martin-Löf randomness are precisely the K-trivials (Hirschfeldt, Nies, and Stephan).
- The Turing bases for Schnorr randomness are precisely the sets that cannot compute $\emptyset'$ (F., Stephan, and Yu).
- The Turing bases for computable randomness have been only partially characterized (Hirschfeldt, Nies, and Stephan; Jockusch, Lerman, Soare, and Solovay).

W-bases for randomness

Definition

A is a W-base for randomness if $A \leq_W Z$ for some A-random set Z.

Trivial results:

- Every jump traceable set is a JT-base for randomness.
- Every K-trivial set is low for randomness and thus an LR-base for randomness.

Question

Are these notions trivial?

JT-bases

Theorem (F., Ng, and Solomon)

Every JT-base for randomness is jump traceable.

JT-bases

Theorem (F., Ng, and Solomon)

Every JT-base for randomness is jump traceable.

Proof:

- Come to my talk at CiE next week!

LR-bases: Background

Proposition (essentially Barmpalias, Lewis, and Stephan)

There is an LR-base that is low for Ω but not K-trivial.

Question

Is being an LR-base equivalent to being K-trivial in the Δ^0_2 degrees?

LR-bases: Background

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Proposition

LR-bases are closed downward under $\leq_{LR}$.

Proposition (Porter)

If $A \leq_{LR} X, Y$ where X and Y are relatively random, then A is an LR-base.

LR-bases: More background

Let's restrict our attention to r.e. LR-bases. We get

$$K\text{-trivials} \subsetneq LR\text{-bases} \subsetneq \text{superlows}.$$

Why?

Proposition (Porter)

There is an r.e. set that is an LR-base and not K-trivial.

Porism

Each LR-base is jump traceable with bound $h(n) = 2^n$, so not every superlow r.e. set is an LR-base.

Bounding the bound function

Theorem (Downey and Greenberg)

Every $\sqrt{\log n}$-jump traceable r.e. set is K-trivial.

This gives us (for r.e. sets)

$$\sqrt{\log n}\text{-jump traceables} \subsetneq LR\text{-bases} \subseteq 2^n\text{-jump traceables.}$$

Question

For which computable functions h are h-jump traceable sets LR-bases?

LR-bases: A bound

Theorem (F., Ng, and Solomon)

Let $\epsilon > 0$. Then for r.e. sets,

$$\frac{n}{(\log n)^{1+\epsilon}}\text{-}\textit{jump traceables} \subseteq \textit{LR-bases} \subseteq n(\log n)^{1+\epsilon}\text{-}\textit{jump traceables}.$$

Furthermore, there is an r.e. LR-base A that is not $n \log n$-jump traceable.

Proof

The proof that

$$\frac{n}{(\log n)^{1+\epsilon}}\text{-jump traceables} \subseteq LR\text{-bases}$$

uses a box promotion strategy. From here on, I will discuss the proof that

$$LR\text{-bases} \subseteq n(\log n)^{1+\epsilon}\text{-jump traceables}$$

by comparing it to attempts to characterize K-triviality in terms of traceability.

Constructing a K-trivial set E

E is K-trivial if every E-r.e. open set of measure < 1 can be covered by an r.e. open set of measure < 1 (Kjos-Hanssen).

Goal: Build V covering the universal E-r.e. open set U^E.

Each positive requirement is assigned some threshold measure δ. If a requirement requires attention, we check the cost of changing E:

$$\text{Is } \mu(U_s^E - U_s^{E \cup \{x\}}) < \delta?$$

If so, we change E at the price of losing δ in V; otherwise, we restrain E and injure the requirement.

Construction of an *LR*-base A

$A \leq_{LR} B$ if and only if for every $\Sigma_1^{0,A}$ class U^A of measure < 1, there is a $\Sigma_1^{0,B}$ class V^B of measure < 1 such that $U^A \subseteq V^B$ (Kjos-Hanssen).

We build a r.e. operator V and a set B such that $U^A \subseteq V^B$ for U^A the universal A-r.e. set of strings of measure < 1 and $\mu(V^B) < 1$.

To make B A-random, we make sure that $B \notin [T^A]$ for some component A of the universal A-Martin-Löf test $\langle T^A \rangle$.

Let α and β be our current approximations to A and B, respectively.

- If a string σ enters U^α, we also need to put it into V^β.
- If we see $[\beta] \subseteq T^\alpha$, we must change to a different β' and enumerate σ into $V^{\beta'}$.
- In general, each σ in U^A costs us $(2^{-|\sigma|})^2$ measure in the functional V^X, so a positive requirement with threshold δ can act if the cost of changing A is at most $\sqrt{\delta}$.

This means that we can tolerate more changes in A compared to E.

We can use this to build an LR-base that isn't K-trivial or that isn't jump traceable at order $n \log n$.

Open questions about *LR*-bases

Question

- *Is there a Δ^0_2 LR-base that is not superlow? (It would have to be low.)*
- *Is there an interesting hyperimmune-free LR-base?*
- *Any question you could ask about K-trivials. (Do they form an ideal?)*

Thank you!