On Friedberg Splits

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Computably Enumerable Sets

- W_e is the eth c.e. set under some nice acceptable uniform standard enumeration of all c.e. sets.
- $W_{e,s} \subseteq \{0, 1, \ldots s\}$.
- A c.e. set R is *computable* iff $\overline{R}$ is also a c.e. set.
- A_0, A_1 is a *split* of A iff $A_0 \sqcup A_1 = A$ iff $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = A$.
- Focus on splits of noncomputable c.e. sets into c.e. sets.
- If $F \subseteq A$ is finite than $F \sqcup (A - F) = A$.
- A split A_0, A_1 is *trivial* if A_0 or A_1 is computable.

Nontrivial Trivial Splits

Lemma

Every noncomputable c.e. set A has an infinite computable subset R.

Then $A = R \sqcup (A \cap \overline{R})$.

Proof.

$A = a_0, a_1, a_2 \ldots$, in the order of enumeration with no repeats. Let $R = \{a_i | (\forall j \leq i)[a_i > a_j]\}$. $n \in R$ iff, for some $i \leq n$, $n = a_i$, and, for all $j < i$, $a_j < a_i$. □

Myhill's Question

Question

Does every noncomputable c.e. set have a nontrivial split?

Theorem (Friedberg)

Yes!

Myhill's question appeared in the Journal of Symbolic Logic in June 1956, Volume 21, Number 2 on page 215 in the "Problems" section of the JSL. This question was the eighth problem appearing in this section. The question about the existence of maximal sets, also answered by Friedberg, was ninth.

Friedberg Splits

Definition

$A_0 \sqcup A_1 = A$ is a *Friedberg Split* of A iff, for all e, if $W_e - A$ is not c.e. then $W_e - A_i$ are also not c.e.

Lemma

A Friedberg split of a noncomputable set is a nontrivial split.

Proof.

Assume A_0 is computable. So $\overline{A_0}$ is a c.e. set.
$\overline{A_0} - A = \overline{A_0} - A_1 = \overline{A}$. So this set is not a c.e. set. But then $\overline{A_0} - A_0 = \overline{A_0}$ must not be c.e. set. Contradiction.

□

This lemma only depends on e such that $W_e - A = \overline{A}$. But which indices are these?

C.e. sets from the enumeration of A

- $W \backslash A = \{x | (\exists s)[x \in W_s \& x \notin A_s]\}$. ($W$ and then maybe A.)
- $W \searrow A = (W \backslash A) \cap A$. ($W$ and then A.)
- $(W \backslash A) = (W - A) \sqcup (W \searrow A)$.
- $(W - A) = (W \backslash A) \sqcup \overline{(W \searrow A)}$
- So if $W - A$ is not a c.e. set then $W \searrow A$ is not computable and hence infinite.

Sufficient to build a Friedberg Split

Lemma

If $A_0 \sqcup A_1 = A$ and, for all e, if $W_e \searrow A$ is infinite then $W_e \searrow A_i$ is infinite, then A_0, A_1 is a Friedberg split of A.

Proof.

Assume $W - A$ is not a c.e. set but $W - A_0$ is a c.e. set. Let $X = W - A_0$. $X - A = W - A$ is not a c.e. set. So $X \searrow A$ is infinite. Therefore $X \searrow A_0$ is infinite. Contradiction. □

Building a Friedberg Split

Theorem (Friedberg)

Every noncomputable set has a Friedberg Split.

Proof.

Use a priority argument to meet the following

$\mathcal{R}_{e,i,k}$: $W_e \searrow A$ is infinite $\Rightarrow (\exists x > k)[x \in A_i]$

□

Corollary

There is a computable total function $f(e) = \langle e_0, e_1 \rangle$ such that if W_e is noncomputable then $W_{f(e_0)}, W_{f(e_1)}$ is a Friedberg split of W_e.

The Motivating Questions

Question

When does a c.e. set have a nontrivial nonFriedberg split?

Question

Is it possible to uniformly split all noncomputable c.e. sets into a nontrivial nonFriedberg split?

$\mathcal{D}$-hhsimple Sets

Definition

- $\mathcal{D}(A) = \{B | B - A$ is a c.e. set$\}$.
- W is *complemented* modulo $\mathcal{D}(A)$ iff there is a c.e. Y such that $W \cup Y \cup A = \omega$ and $(W \cap Y) - A$ is a c.e. set. (Drop modulo $\mathcal{D}(A)$.)
- A is *$\mathcal{D}$-hhsimple* iff, for every W, if $A \subseteq W$, W is complemented.
- A complemented W is 0 (modulo $\mathcal{D}(A)$) iff $W - A$ is a c.e. set.
- A complemented W is 1 (modulo $\mathcal{D}(A)$) iff $Y - A$ is a c.e. set (the Y from above). In this case, WLOG $Y \cap A = \emptyset$.
- A is *$\mathcal{D}$-maximal* iff for every W, if $A \subseteq W$, W is complemented and either 0 or 1.

$\mathcal{D}$-maximals Sets

Lemma (Cholak, Downey, Herrmann)

All nontrivial splits of a $\mathcal{D}$-maximal set A are Friedberg.

Proof.

Assume that $W - A$ is not a c.e. set (So W is 1). Then, for some Y, $W \cup A \cup Y =^* \omega$ and $Y \cap A = \emptyset$. If $W - A_0$ is c.e. then $A_0 \sqcup ((W - A_0) \cup A_1 \cup Y) =^* \omega$. So A_0 is computable. Contradiction. □

There are Nontrivial NonFriedberg Splits

- Let R be an infinite, coinfinite computable set. Let R_K be a noncomputable c.e. subset of R.
- Similarly let $\overline{R}_K$ be a noncomputable c.e. subset of $\overline{R}$.
- $R_K \sqcup \overline{R}_K = A$ is a nontrivial nonFriedberg split of A.
- $R - R_K$ is not a c.e. set but $\overline{R} - R_K = \overline{R}$ is a c.e. set.
- Here all 3 sets were built simultaneously. We need both A and R to construct the split.

A More Difficult Example

Theorem

There is split A_0, A_1 of an r-maximal set A such that the split is nontrivial and, for all e, either $W - A_o$ is a c.e. set or there is a D with $D \cap A_0 = \emptyset$ and $A \cup D \cup W =^* \omega$.

So A_0 is $\mathcal{D}$-maximal but there are no restrictions on A_1.

Proof.

Sorry, some other talk. But again all 3 sets are built simultaneously. □

The Kummer and Herrmann Splitting Theorem

Theorem (Kummer and Herrmann)

If $A \subseteq X$ is noncomplemented modulo $\mathcal{D}(A)$ then there are X_0 and X_1 such that X_i is noncomplemented and $A \subseteq X_0 \sqcup X_1 = X$.

Corollary

For all noncomputable non-$\mathcal{D}$-maximal A, there are disjoint X_0 and X_1 such that X_i is noncomplemented and $A \subseteq X_0 \sqcup X_1$.

Proof.

The above theorem applies when A is not $\mathcal{D}$-hhsimple. Otherwise A must have a superset W which is not 0 or 1. So it's complement Y is also not 0 or 1. Let $X_0 = W \backslash Y$ and $X_1 = Y \backslash W$. □

Splits of non-$\mathcal{D}$-maximal Sets

Theorem (Shavrukov)

Let A be not $\mathcal{D}$-maximal and not computable. Then A has a nontrivial nonFriedberg split.

Proof.

There are X_0, X_1 such that they are noncomplemented and $A \subseteq X_0 \sqcup X_1$. $X_i - A$ is not a c.e. set (otherwise X_i is 0 and complemented). So $X_i \cap A$ is not computable and $X_i - (X_{\bar{i}} \cap A) = X_i$ is a c.e. set. Hence $X_0 \cap A, X_1 \cap A$ is a nontrivial nonFriedberg split. □

The Motivating Questions, Again

Question

When does a c.e. set have a nontrivial nonFriedberg split?

Theorem (Shavrukov)

All of A's nontrivial splits are Friedberg iff A is $\mathcal{D}$-maximal.

Question

Is it possible to uniformly split all noncomputable c.e. sets into a nontrivial nonFriedberg split?

No.

Question

Is it possible to uniformly split all non $\mathcal{D}$-maximal sets into a nontrivial nonFriedberg split?

Still no.

No Uniform Nontrivial NonFriedberg Splits

Theorem (Cholak)

For every computable f there is an e such that W_e is not computable and if $f(e) = \langle e_0, e_1 \rangle$ then either

- W_{e_0}, W_{e_1} *is not a split of* W_e*,*
- W_{e_0}, W_{e_1} *is a trivial split of* W_e*, or*
- W_{e_0}, W_{e_1} *is a Friedberg split of* W_e *and* W_e *is not* $\mathcal{D}$*-maximal.*

The Construction Viewed from $\mathbf{0}''$

Build $A = W_e$ via the recursion theorem. Assume that $f(e) = \langle e_0, e_1 \rangle$. Build infinite computable pairwise disjoint sets such that

$$\sharp \qquad (\forall i)[W_i \subseteq \bigsqcup_{j \le i} R_j \text{ or } W_i \cup A \cup \bigsqcup_{j \le i} R_j =^* \omega]$$

Inside each R_i try to build A to be maximal via Friedberg's maximal set construction. So A is not computable. Assume that $W_{e_0} = A_0, W_{e_1} = A_1$ is a split (otherwise done). Now in R_i ask is

$$\star \qquad A_0 \cap R_i \text{ infinite?}$$

If no, then we want to focus the construction of A at R_i. For $j < i$ dump every ball possible into A. For $j > i$, put no balls into A. So A is only noncomputable inside R_i and hence A_0, A_1 is a trivial split. Similarly, if $A_1 \cap R_i$ is finite.

The Verification

Assume we have postive answers to $\star$ for e_0 and e_1. So A is maximal inside each R_i. The R_i modulo $\mathcal{D}(A)$ witness that A is not $\mathcal{D}$-maximal. So A has a nontrivial nonFriedberg split. Locally inside each R_i, our split A_0, A_1 is Friedberg. We must show globally that A_0, A_1 is a Friedberg split. Consider W_i and assume $W_i - A$ is not a c.e. set. Now $\sharp$ holds. If the first clause of $\sharp$ holds, then W_i is handled locally inside R_j for $j \leq i$ and $W_i - A_l$ is not a c.e. set. Otherwise $R_{i+1} - A \subseteq W_i$. This implies that $(W_i - A_l) \cap R_{i+1}$ is not a c.e. set.