Solovay Functions and the No-gap Phenomena

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infinitely often upper bound of K and C

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- $\forall x \mathrm{C}(x) \leq^{+} |x|$.
- $\exists^{\infty} x\ \mathrm{C}(x) \geq^{+} |x|$.

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Definition

A function g is a *Solovay function* if g is computable and it holds that

1. $\forall x[\mathrm{K}(x) \leq^+ g(x)]$
2. $\exists^\infty x[\mathrm{K}(x) \geq^+ g(x)]$

A function g is a *weak Solovay function* if g is right-c.e. and satisfies both 1 and 2.

An equivalent characterization for Solovay functions

Theorem

Let $f : \mathbb{N} \to \mathbb{N}$ be a right-c.e. function. Then f is an upper bound of K iff $\sum_n 2^{-f(n)}$ is finite.

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Theorem (Bienvenu and Downey, 2009)

Let $f : \mathbb{N} \to \mathbb{N}$ *be a right-c.e. function. Then* f *is a weak Solovay function* $\Leftrightarrow$ $\sum_n 2^{-f(n)}$ *is finite and is a Martin-Löf random real.*

K-triviality and Solovay functions

Definition

A sequence A is K-*trivial* if $\forall n\ \mathrm{K}(A \upharpoonright n) \leq^{+} \mathrm{K}(n)$.

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Actually, we can replace $\mathrm{K}(n)$ in the definition by any weak Solovay function.

Theorem (Bienvenu, Merkle and Nies, 2011)

If g is a (weak) Solovay function, then $(*)$ a sequence A is K-trivial iff $\forall n\, \mathrm{K}(A \upharpoonright n) \leq^+ g(n)$.

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If g is a (weak) Solovay function, then $(*)$ a sequence A is K-trivial iff $\forall n\, \mathrm{K}(A \upharpoonright n) \leq^+ g(n)$.

And $(*)$ turns out to be a characterization of Solovay function among all right-c.e. functions.

Theorem (Bienvenu, Downey, Nies and Merkle, 2015)

If g is a computable (right-c.e.) function such that for any sequence A, A is K-trivial iff $\forall n\, \mathrm{K}(A \upharpoonright n) \leq^+ g(n)$, then g is a (weak) Solovay function.

Gács-Miller-Yu theorem and Solovay functions

Theorem (Gács-Miller-Yu)

A sequence A is Martin-Löf random iff for all $n \in \omega$, $\mathrm{C}(A \upharpoonright n) \geq^+ n - \mathrm{K}(n)$.

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Let g be a computable (right-c.e.) function such that for any sequence A, A is Martin-Löf random iff $\forall n\, \mathrm{C}(A \upharpoonright n) \geq^{+} n - g(n)$, *then g is a (weak) Solovay function.*

Weak lowness for K and Solovay functions

Definition

- A sequence A is *weakly low for* K if $\exists^{\infty} n \mathrm{K}^{A}(n) \geq \mathrm{K}(n)$;
- A sequence A is *low for* Ω if Ω is Martin-Löf -random relative to A.

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Miller first showed that these two lowness are equivalent, while Bienvenu noticed a simple proof using Solovay function:

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Miller first showed that these two lowness are equivalent, while Bienvenu noticed a simple proof using Solovay function:

- Function K is right-c.e., it is also right-c.e. relative to A.
- And K is also an upper bound for K^A up to an additive constant.
- By definition, A is weakly low for K iff K is a weak Solovay function relative to A.
- Relativizing the equivalent characterization of Solovay function, K is a weak Solovay function relative to A iff $\Omega_\mathrm{K} = \sum_n 2^{-\mathrm{K}(n)}$ is Martin-Löf random relative to A.
- So A is weakly low for K iff A is low for Ω.

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If g is a weak Solovay function, then a sequence A is weakly low for K iff $\exists^\infty n \mathrm{K}^A(n) \geq^+ g(n)$.

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- One direction is trivial.
- A is weakly low for K, then it is low for Ω.
- $\Omega_g = \sum_n 2^{-g(n)}$ is 1-random and left-c.e., then by Kučera-Slaman Theorem, it is Ω-like.
- Then Ω_g is 1-random relative to A.
- By relativization, $\exists^\infty n \mathrm{K}^A(n) \geq^+ g(n)$.

Weak lowness for K and Solovay functions

Theorem

Let g be a right-c.e. function such that for any sequence A, A is weakly low for K iff $\exists^\infty n \mathrm{K}^A(n) \geq^+ g(n)$, then g is a weak Solovay function.

Weak lowness for K and Solovay functions

Theorem

Let g be a right-c.e. function such that for any sequence A, A is weakly low for K iff $\exists^\infty n \mathrm{K}^A(n) \geq^+ g(n)$, then g is a weak Solovay function.

- For all sequence A, $\forall n\, \mathrm{K}^A(n) \leq^+ \mathrm{K}(n)$.
- If for some sequence A, $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ g(n)$, then $\exists^\infty n\, \mathrm{K}(n) \geq^+ g(n)$.
- If g is not an upper bound of K, then $\sum_n 2^{-g(n)} = \infty$.
- For all A, $\sum_n 2^{-\mathrm{K}^A(n)} < \infty$, $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ g(n)$.

2-randomness and Solovay functions

Theorem (Miller)

A set A is 2-random iff $\exists^\infty n \mathrm{K}(A \upharpoonright n) \geq^+ \mathrm{K}(n) + n$.

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Theorem

If g is a weak Solovay function, then a sequence A is 2-random iff $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$.

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If g is a weak Solovay function, then a sequence A is 2-random iff $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$.

- A is 2-random iff A is 1-random and low for Ω.
- A is 1-random, by Ample Excess Lemma, $\forall n\, \mathrm{K}^A(n) \leq^+ \mathrm{K}(A \upharpoonright n) - n$.
- A is low for Ω, by previous result, $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ g(n)$.
- Thus, $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + g(n)$.

2-randomness and Solovay functions

Theorem

If f is a right-c.e. function, and for any sequence A, A is 2-random iff $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$, then f is a weak Solovay function.

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If f is a right-c.e. function, and for any sequence A, A is 2-random iff $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$, then f is a weak Solovay function.

- For all sequence A, $\forall n\, \mathrm{K}(A \upharpoonright n) \leq^+ n + \mathrm{K}(n)$.
- If for some sequence A, $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$, then $\exists^\infty n\, \mathrm{K}(n) \geq^+ f(n)$.
- If g is not an upper bound of K, then for all A, $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ f(n)$.
- By Ample Excess Lemma, then all 1-random sequences A, $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$.

Infinitely often K-triviality and Solovay functions

Definition

A sequence A is *infinitely often* K*-trivial* if there are infinitely many point n such that $\mathrm{K}(A \upharpoonright n) \leq^+ \mathrm{K}(n)$.

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A sequence A is *infinitely often* K*-trivial* if there are infinitely many point n such that $\mathrm{K}(A \upharpoonright n) \leq^+ \mathrm{K}(n)$.

It seems very promising that in the definition the function $\mathrm{K}(n)$ can be replaced by arbitrary Solovay function, but we will see that it is false.

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Theorem

There is a Solovay function f that for some sequence A there are infinitely many point n such that $\mathrm{K}(A \upharpoonright n) \leq^+ f(n)$ but A is not infinitely often K-trivial

Infinitely often K-triviality and Solovay functions

Proof.

- Suppose f is a Solovay function, define f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} f(x) & \text{if } x \text{ is odd} \\ 2x & \text{if } x \text{ is even} \end{cases} \quad f_2(x) = \begin{cases} 2x & \text{if } x \text{ is odd} \\ f(x) & \text{if } x \text{ is even} \end{cases}$$

- $\forall x\, \mathrm{K}(x) \leq^+ 2|x| \leq 2x$, and $\forall x\, \mathrm{K}(x) \leq^+ f(x)$, then $\forall x\, \mathrm{K}(x) \leq^+ f_1(x)$ and $\mathrm{K}(x) \leq^+ f_2(x)$.
- As there are infinitely many x such that $\mathrm{K}(x) \geq^+ f(x)$. then at least for one of $f_i (i = 0, 1)$, there are infinitely many x such that $\mathrm{K}(x) \geq^+ f_i(x)$.
- Suppose $i = 1$, then f_1 is a Solovay function.
- For any sequence A, for all even number n, $\mathrm{K}(A \upharpoonright n) \leq^+ 2n = f_1(n)$.

□

Infinitely often $\mathbb{K}$-triviality and Solovay functions

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Theorem

For any function f which goes to infinity, there exists a sequence A such that A is not infinitely often K-trivial but $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \leq^+ \mathrm{K}(n) + f(n)$.

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For any function f which goes to infinity, there exists a sequence A such that A is not infinitely often K-trivial but $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \leq^+ \mathrm{K}(n) + f(n)$.

That's to say, among all right-c.e. functions which are upper bounds of K, if for any sequence A, A is infinitely often K-trivial iff $\exists^\infty n \mathrm{K}(A \upharpoonright n) \leq^+ g(n)$, then g is a weak Solovay function.

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But whether all computable (right-c.e.) functions which make the equivalence ture should be (weak) Solovay functions is still open.

Summary

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Let g be any weak Solovay function, the following assertions are ture.

1. $\sum_n 2^{-g(n)}$ is a Martin-Löf random real.
2. A sequence A is K-trivial iff $\forall n\, \mathrm{K}(A \upharpoonright n) \leq^+ g(n)$.
3. A sequence A is Martin-Löf random iff $\forall n\, \mathrm{C}(A \upharpoonright n) \geq^+ n - g(n)$.
4. A sequence A is weakly low for K, iff $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ g(n)$.
5. A sequence A is 2-random, iff $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + g(n)$.

What's more, among all right-c.e. functions the respective assertion is ture exactly for the Solovay functions.

The no-gap phenomena

Theorem

Suppose f is a right-c.e. function, the following are equivalent:

1. $\forall x[\mathrm{K}(x) \leq^+ f(x)]$;
2. $\sum_n 2^{-f(n)} < \infty$;
3. *If A is* K*-trivial, then* $\forall n\, \mathrm{K}(A \upharpoonright n) \leq^+ f(n)$;
4. *If A is 1-random, then* $\forall n\, \mathrm{C}(A \upharpoonright n) \geq^+ n - f(n)$.
5. *If* $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ f(n)$*, then A is weakly low for* K;
6. *If* $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$*, then A is 2-random;*

The no-gap phenomena

Theorem

Suppose f is a right-c.e. function, and is an upper bound for K*, the following are equivalent:*

1. $\exists^\infty x[\mathrm{K}(x) \geq^+ f(x)]$;
2. $\sum_n 2^{-f(n)}$ *is 1-random;*
3. *If* $\forall n\, K(A \upharpoonright n) \leq^+ f(n)$*, then* A *is* K*-trivial;*
4. *If* A *is weakly low for* K*, then* $\exists^\infty n\, \mathrm{K}^A(n) \geq^+ f(n)$*;*
5. *If* A *is 2-random, then* $\exists^\infty n\, \mathrm{K}(A \upharpoonright n) \geq^+ n + f(n)$*;*
6. *If* $\forall n\, \mathrm{C}(A \upharpoonright n) \geq^+ n - f(n)$*, then* A *is 1-random.*

The no-gap phenomena

In the proof of our previous theorems, we proved the following so-called “no-gap" theorems:

No-gap

There is no function $h : \mathbb{N} \mapsto \mathbb{N}$ which tends to infinity and such that:

1. $\mathrm{C}(A \upharpoonright n) \geq^{+} n - \mathrm{K}(n) - h(n) \implies A$ is Martin-Löf randomn;
2. $\mathrm{K}(A \upharpoonright n) \leq^{+} \mathrm{K}(n) + h(n) \implies A$ is infinitely often K-trivial;

For K-triviality, George and Charlotte showed that there is no Δ^0_2 “gap”, but Martijn and George showed there does exist a Δ^0_3 “gap”.