Weihrauch-completeness for layerwise computability[1]

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[1] Joint work with George Davie & Willem Fouché (UNISA).

Outline

Layerwise computability

Fix a universal Martin-Löf test $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$.

Definition

A (multivalued) function $f : \mathrm{MLR} \rightrightarrows \mathbf{X}$ is layerwise computable w.r.t. $\mathcal{U}$, iff there exists a computable partial function $F :\subseteq \mathbb{N} \times \mathrm{MLR} \to \mathbf{X}$ such that whenever $p \notin U_n$ then $F(n, p) \in f(p)$.

Theorem (Hölzl & Shafer)

Layerwise computability does depend on the choice of $\mathcal{U}$ in general, but all optimal Martin-Löf tests yield the same class.

More extended computability notions

Definition

A finitely-revising machine is a Type-2 machine with the extra capability to erase its output and restart writing it, to be used finitely many times during the computation. A function is computable with finitely many mindchanges, if there this a finitely-revising machine computing it.

Definition

A non-deterministic Type-2 machine with advice space $\mathbf{Z}$ computes a multivalued function $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ as follows:

1. On input $x \in \mathbf{X}$, guess some $z \in \mathbf{Z}$.
2. Either: Halt and reject the guess.
3. Or: Run indefinitely, and output some $y \in f(x)$.

Such that for any $x \in \mathbf{X}$ there is some $z \in \mathbf{Z}$ leading to case 3.

Connections

Observation (Brattka, de Brecht & P.)

Finitely revising machines and non-deterministic machines with advice space $\mathbb{N}$ are equivalent.

Observation

Any layerwise computable function is computable by non-deterministic machine with advice space $\mathbb{N}$.

Represented spaces and computability

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ a surjective partial function.

Definition

$F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a realizer of $f : \mathbf{X} \rightrightarrows \mathbf{Y}$, iff $\delta_Y(F(p)) \in f(\delta_X(p))$ for all $p \in \delta_X^{-1}(\text{dom}(F))$.

$$\begin{array}{ccc} \mathbb{N}^\mathbb{N} & \xrightarrow{F} & \mathbb{N}^\mathbb{N} \\ \downarrow \delta_X & & \downarrow \delta_Y \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

Definition

$f : \mathbf{X} \rightrightarrows \mathbf{Y}$ is called computable (continuous), iff it has a computable (continuous) realizer.

Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, such that $K\langle \mathrm{id}_{\mathbb{N}^\mathbb{N}}, GH\rangle$ is a realizer of f for every realizer G of g.

Theorem (Brattka & Gherardi 2011, P. 2010)

$\mathfrak{W}$ is a distributive lattice. The cartesian product $\times$ is an operation on $\mathfrak{W}$.

Theorem (Higuchi & P. 2013)

For $A \subseteq \mathbb{N}^\mathbb{N}$, let $d_A : A \to \{0\}$. Then $d_. : \mathfrak{M}^{op} \to \mathfrak{W}$ is a lattice embedding.

The motivation

1. Identify a theorem

$$\forall x \in \mathbf{X}\ \exists y \in \mathbf{Y} \,.\, D(x) \Rightarrow\ T(x, y)$$

   with the multi-valued function $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $\mathrm{dom}(T) = D$ obtained by Skolemization.
2. Then compare theorems via Weihrauch-reducibility to learn about their *constructive content*.

Similar spirit as (constructive) reverse mathematics, but:

Theorem (Higuchi & P. 2013)

$\mathfrak{W}$ is not a Brouwer algebra.

The degree of $C_{\mathbb{N}}$

Lemma

The following are Weihrauch equivalent:

1. $C_{\mathbb{N}} :\subseteq \mathcal{A}(\mathbb{N}) \rightrightarrows \mathbb{N}$ *be defined via* $n \in C_{\mathbb{N}}(A)$ *iff* $n \in A$
2. $UC_{\mathbb{N}}$, *defined via* $UC_{\mathbb{N}} = (C_{\mathbb{N}})|_{\{A \in \mathcal{A}(\mathbb{N}) \mid |A| = 1\}}$
3. $\min_{\mathcal{A}} :\subseteq \mathcal{A}(\mathbb{N}) \to \mathbb{N}$
4. $\max_{\mathcal{O}} :\subseteq \mathcal{O}(\mathbb{N}) \to \mathbb{N}$
5. $\text{Bound} :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbb{N}$, *where* $n \in \text{Bound}(U)$ *iff* $\forall m \in U\ n \geq m$.

Weihrauch-completeness for layerwise-computability

Definition

Let $\mathrm{LAY}_{\mathcal{U}} : \mathrm{MLR} \rightrightarrows \mathbb{N}$ be defined via $n \in \mathrm{LAY}_{\mathcal{U}}(p)$ iff $p \notin U_n$.
Let $\mathrm{rd}_{\mathcal{U}} : \mathrm{MLR} \to \mathbb{N}$ be defined via
$\mathrm{rd}_{\mathcal{U}}(p) = \min\{n \in \mathbb{N} \mid p \notin U_n\}$.

Observation

$LAY_{\mathcal{U}}$ is layerwise computable w.r.t. $\mathcal{U}$. Whenver $f : MLR \rightrightarrows \mathbf{X}$ is layerwise computable w.r.t. $\mathcal{U}$, then $f \leq_W LAY_{\mathcal{U}}$.

- If f is layerwise-computable and $f \equiv_W \mathrm{LAY}_{\mathcal{U}}$, call f Weihrauch-complete for layerwise computability.
- The problems that are Weihrauch-complete for layerwise computability are the *most non-computable layerwise-computable problems*.

The main theorem

Theorem

$LAY_{\mathcal{U}} \equiv_W rd_{\mathcal{U}} \equiv_W C_{\mathbb{N}} \times d_{MLR}$

Proof.

$\mathrm{LAY}_{\mathcal{U}} \leq_{\mathrm{W}} \mathrm{rd}_{\mathcal{U}}$ Trivial.

$\mathrm{rd}_{\mathcal{U}} \leq_{\mathrm{W}} \min_{\mathcal{A}} \times d_{\mathrm{MLR}}$ We have a random sequence available as input for d_{MLR}, and the presence of this degree does not matter further. Note that given p we can compute $\{n \mid p \notin U_n\} \in \mathcal{A}(\mathbb{N})$.

□

Proof continued

Proof.

Bound $\times d_{\mathrm{MLR}} \leq_{\mathrm{W}} \mathrm{LAY}_{\mathcal{U}}$ The input is an enumeration of some finite set $I \subset \mathbb{N}$ (which we may safely assume to be an interval) and a random sequence p. Let w be the current prefix of the output (i.e. the input to $\mathrm{LAY}_{\mathcal{U}}$). If we learn that $n \in I$, we consider $w0^{\mathbb{N}}$. As this is not random and $\mathcal{U}$ is universal, we know that $w0^{\mathbb{N}} \in U_n$. As U_n is open, there is some – effectively findable – $k \in \mathbb{N}$ such that $w0^k\{0,1\}^{\mathbb{N}} \subseteq U_n$. We proceed to amend the current output to $w0^k$, and then start outputting p (until we potentially learn $n+1 \in I$.

As I is finite, the output q will have some tail identical to p, and thus is Martin Löf random. By construction, whenever $n \in I$, then $q \in U_n$, thus if $b \in \mathrm{LAY}_{\mathcal{U}}(q)$ then $b \in$ Bound(p).

□

Corollaries

- $\mathrm{LAY} <_W \mathrm{C}_{\mathbb{N}}$
- $\mathrm{LAY} \times \mathrm{LAY} \equiv_W \mathrm{LAY}$ and $\mathrm{LAY} \star \mathrm{LAY} \equiv_W \mathrm{LAY}$
- $\mathrm{LAY} \star \mathrm{C}_{\mathbb{N}} \equiv_W \mathrm{C}_{\mathbb{N}} \star \mathrm{LAY} \equiv_W \mathrm{LAY}$
- $\mathrm{LAY} <_W \widehat{\mathrm{LAY}} \equiv_W \lim \times d_{\mathrm{MLR}}$
- $\mathrm{LAY} <_W \mathrm{LAY}^* \equiv_W \mathrm{id}_{\mathbb{N}^{\mathbb{N}}} + \mathrm{LAY} <_W \mathrm{C}_{\mathbb{N}}$
- If $f \leq_W \mathrm{C}_{\mathbb{N}}$ for $f :\subseteq \mathrm{MLR} \rightrightarrows \mathbf{Y}$, then $f \leq_W \mathrm{LAY}$.

More consequences

Corollary

The following are equivalent for $f :\subseteq MLR \to \mathbf{Y}$ for a computable metric space $\mathbf{Y}$:

1. *f is effectively Δ^0_2-measurable.*
2. *f is Π^0_1-piecewise computable.*
3. *$f \leq_W LAY$.*

Proof.

By combining the computable Jayne-Rogers theorem (P. & de Brecht 2014) with the main theorem. □

Complex oscillations

Definition

The complex oscillations CO are the Martin-Löf random elements of $\mathcal{C}_0([0,1],\mathbb{R})$ equipped with the Wiener measure. Let computable $\eta : \mathrm{MLR} \to \mathbb{R}$ induce the normal distribution $\mathcal{N}(0,1)$ on $\mathbb{R}$.

Definition

We define the function $\Phi : \mathrm{MLR} \to \mathrm{CO}$ by recursively providing the values $\Phi(\alpha)$ takes on dyadic rationals, and extending it continuously to the interval. Let $\alpha = \langle \alpha_0, \alpha_1, \ldots, \alpha_{jn}, \ldots \rangle$, where $n \leq 2^j$. Then we define:

1. $\Phi(\alpha)(1) := \eta(\alpha_0)$
2. $\Phi(\alpha)(\frac{1}{2}) := \frac{1}{2}\left(\eta(\alpha_0) + \eta(\alpha_1)\right)$
3. $\Phi(\alpha)(\frac{2n+1}{2^{j+1}}) := \frac{1}{2}\left(2^{-j/2}\eta(\alpha_{jn}) + \Phi(\alpha)(\frac{n+1}{2^j}) + \Phi(\alpha)(\frac{n}{2^j})\right)$

Theorem (Davie & Fouché)

Φ *is a layerwise computable bijection with computable inverse.*

The completeness result

Theorem

$\Phi \equiv_W LAY$

Lemma

Given $k \in \mathbb{N}$ and $v \in \{0,1\}^*$ we can compute some $w \in \{0,1\}^*$ such that for all $\alpha \in MLR$ we find that $k < \sup_{t \in [0,1]} \Phi(vw\alpha)(t)$.

Law of the iterated logarithm

Definition

Let $\mathrm{LIL} : \mathrm{MLR} \rightrightarrows \mathbb{N}$ be defined via $N \in \mathrm{LIL}(\alpha)$ iff:

$$\forall n \geq N \quad |\sum_{i=0}^{n-1}(2\alpha(i) - 1)| < \sqrt{2n \log \log n}$$

Theorem

LIL $\equiv_W$ *LAY*.

Lemma

Given $N \in \mathbb{N}$ *and* $u \in \{0,1\}^*$ *we can compute some* $v \in \{0,1\}^*$ *such that* $|uv| > N$ *and* $|\sum_{i=0}^{|uv|-1}(2(uv)(i) - 1)| > \sqrt{2|uv| \log \log |uv|}$.

Birkhoff's theorem

Definition

Let $S : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the usual shift-operator, and $\pi_1 : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ be the projection to the first bit. Let $\mathrm{Birkhoff} : \mathrm{MLR} \times \mathbb{N} \rightrightarrows \mathbb{N}$ be defined via $N \in \mathrm{Birkhoff}(p,k)$ iff $\forall n \geq N$ we find that:

$$|\left(\frac{1}{n+1}\sum_{i=0}^{n} \pi_1(S^i(p))\right) - \frac{1}{2}| < 2^{-k}$$

Theorem

$\mathrm{Birkhoff} \equiv_W \mathit{LAY}$

Proof ingredient

Lemma

Given $u \in \{0,1\}^*$ and $k, N \in \mathbb{N}$, $k > 0$, we can compute some $v \in \{0,1\}^*$ such that $|uv| \geq N$ and:

$$| \left(\frac{1}{|uv|} \sum_{i=0}^{|uv|-1} \pi_1(S^i(uv)) \right) - \frac{1}{2} | > 2^{-k}$$

Hitting times

Definition

Let $\mathcal{A}_{\lambda>0}(\{0,1\}^{\mathbb{N}})$ be the restriction of $\mathcal{A}(\{0,1\}^{\mathbb{N}})$ to sets of positive Lebesgue measure. Let $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the usual shift-operator. Define $\mathrm{HittingTime}_{\mathcal{A}} : \mathrm{MLR} \times \mathcal{A}_{\lambda>0}(\{0,1\}^{\mathbb{N}}) \to \mathbb{N}$ be defined via $\mathrm{HittingTime}_{\mathcal{A}}(p, A) = \min\{n \in \mathbb{N} \mid T^n(p) \in A\}$.

Theorem (Kučera)

$\mathrm{HittingTime}_{\mathcal{A}}$ *is well-defined.*

Theorem

$\mathrm{HittingTime}_{\mathcal{A}} \equiv_W LAY$*, but not even* $\mathrm{HittingTime}_{\mathcal{A}}(\cdot, U_{100}^C)$ *is layerwise computable.*

Some last minute-additions

- Finding the suitable *n* from the multiple recurrence theorem for Martin-Löf randoms is Weihrauch-equivalent to LAY (but not layerwise computable).
- Computing the time-reversal of a Brownian motion on $[0, \infty)$ should be Weihrauch-reducible to LAY (but what about the other direction)?

Some open questions

- Investigate further layerwise-computable problems.
- Is there a (natural) problem which is non-computable, layerwise computable and strictly below LAY?

Reference

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