# How to Gamble Against All Odds

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# Preface

- starting with an algorithmic randomness problem
- transformed to a similar game, without computability

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- For $A \subset \mathbb{R}_+$, $A$-valued random if limiting wagers to $A$ $\forall \sigma \ |M(\sigma \mathrm{h}) - M(\sigma)| \in A$

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- $A = \{1, 2\}$, $B = \{1/2, 1\}$
- Countably many $B$-strategies

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- "not evade" is a preorder
- "not evade each other" is an equivalence relation

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- Either the opponent is bankrupt, or he stops betting.
- (fix cheating) casino chooses tails to signal phase change

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- still not sufficient in general; but is sufficient for some classes of $A, B$

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**Theorem**

*If $A$ is bounded, $0 \notin \overline{B \setminus \{0\}}$, and $B$ does not contain a multiple of $A$, then $A$ evades $B$.*

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- Evens evade odds, but not vice versa.

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- Compare with previous example; $P$ unbounded

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- For any $M \geq 0 \quad q_M(x) = \max(P(x) \cap [0, M]) = 0$

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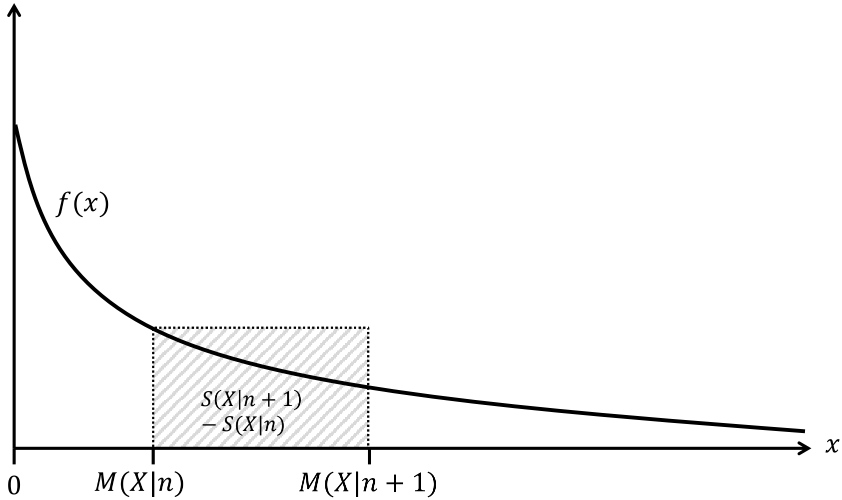
- Attention to the smallest index active

# Not evade



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- The results apply to all variants
- Also for different $B_i$ to each opponent

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- Does $\{1/n : n \in \mathbb{N}\}$ evade $\{1/2^n : n \in \mathbb{N}\}$ ?
- Probabilistic martingales

The End

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